1–f1 The Basics

Whole numbers 1, 2, 3, 4,... are called natural numbers because they are used to count whole items like chickens and cows.

If \( x \) and \( y \) are two natural numbers and \( x \) can be divided by \( y \), we say \( y \) is a factor of \( x \). We also say that \( x \) is a multiple of \( y \). Hence 3 is a factor of 15 and that 15 is a multiple of 3. A number \( x \) is always the multiple and a factor of itself.

Example A. List the factors and the multiples of 12.
The factors of 12: 1, 2, 3, 4, 6, and 12.
The multiples of 12: 12=1\times 12, 24=2\times 12, 36=3\times 12, 48, 60, etc...

[Your Turn: List the factors and the multiples of 18.]

A prime number is a natural number that can not be divided evenly into smaller numbers (except as 1’s), that is to say, a prime number only has 1 and itself as factors. The numbers 2, 3, 5, 7, (not 9), 11, 13,... are prime numbers. Even numbers are not prime so prime numbers must be odd (except for 2). The number 1 is not a prime number.

To factor a number \( x \) means to break down \( x \) as a product, i.e. to write \( x \) as \( a\cdot b\cdot c\cdot... \) e.g. 12 can be factored as
\[ 12 = 2\cdot 6 = 3\cdot 4 = 2\cdot 2\cdot 3. \]
We say the factorization is complete if all the factors are prime numbers. Hence 12=2\cdot 2\cdot 3 is factored completely, 12=2\cdot 6 is not complete because 6 is not prime. Just as in chemistry that every chemical is a group of bonded elements, every number is the product of a group of prime numbers.

Exponents

To simplify writing repetitive multiplication, we write \( 2^2 \) for 2\cdot 2, we write \( 2^3 \) for 2\cdot 2\cdot 2, we write \( 2^4 \) for 2\cdot 2\cdot 2\cdot 2 and so on.

We write \( x\cdot x\cdot x\cdot...\cdot x \) as \( x^N \) where \( N \) is the number of \( x \)'s multiplying to itself. \( N \) is called the exponent or the power of \( x^N \).

Example B. Calculate.

\[ 3^2 = 3\cdot 3 = 9 \quad \text{(note that 3\cdot 2 = 6)} \]
\[ 4^3 = 4\cdot 4\cdot 4 = 64 \quad \text{(note that 4\cdot 3 = 12)} \]

[Your Turn: Calculate \( 2^4, 3^4, 4^4, \text{ and } 2\cdot 4, 3\cdot 4, 4\cdot 4 \).]
Numbers that are factored completely may be written using the exponential notation. Hence, factored completely, $12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$, $200 = 8 \cdot 25 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$.

Each number has a unique form when its completely factored into prime factors and arranged in order. This is analogous to chemistry where each chemical has a unique chemical composition of basic elements such as $H_2O$ for water.

Larger numbers may be factored using a vertical format.

Example C. Factor 144 completely. (Vertical format)

```
  144
   \  / \
   12  12
  / \ / \ \\
 3  4 3 4
 /  /  /  \\
2  2  2  2
```

Gather all the prime numbers at the end of the branches we have $144 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 3^2 \cdot 2^4$.

Note that we obtain the same answer regardless how we factor at each step.

Your Turn: Factor 72 completely.

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**Basic Laws**

Of the four arithmetic operations $+,-,\times$, and $\div$, $+,$ and $\times$ behave nicer than $-$ or $\div$. We often take advantage of the this nice “behavior” of the addition and multiplication operations to get short cuts. However, be sure you don’t mistakenly apply these short cuts to $-$ and $\div$.

**Associative Law for Addition and Multiplication**

$(a + b) + c = a + (b + c)$  
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Note: We are to perform the operations inside the “( )” first. For example, $(1 + 2) + 3 = 3 + 3 = 6$, which is the same as $1 + (2 + 3) = 1 + 5 = 6$.

Subtraction and division are not associative. For example, $(3 - 2) - 1 = 0$ is different from $3 - (2 - 1) = 2$. So if you were to total the money in your pockets, it does not matter which two pockets you add first.
Commutative Law for Addition and Multiplication

a + b = b + a
a · b = b · a

For example, 3·4 = 4·3 = 12.
Subtraction and division don’t satisfy commutative laws.
For example: 2 ÷ 1 ≠ 1 ÷ 2.

From the above laws, we get the following important facts.
**When adding, the order of addition doesn’t matter.**
Hence to take advantages of this when adding a list numbers, it’s easier to add the ones that add to multiples of 10 first.

Example D.

\[
\begin{align*}
14 + 3 + 16 + 8 + 35 + 15 & = 30 + 11 + 50 = 91
\end{align*}
\]

**When multiplying, the order of multiplication doesn’t matter.**
When multiplying many numbers, always multiply them in pairs first. This is useful for raising exponents.

Example E.

\[
\begin{align*}
3^6 & = 3·3·3·3·3·3 \\
& = 9 · 9 · 9 \\
& = 81 · 9 = 729
\end{align*}
\]

Because the order of the additions (or multiplications) may be scrambled, this offers a strategy to check our answers—**do it in two different ways.**
*If both yield the same answer, it’s likely the answer is correct. If the two outcomes are different, then some thing went wrong, find it and correct it.*
The Distributive Law

There are two $10$-bills and three $5$-bills in a box. So there is the total of $2 \times 10 = 20$ plus $3 \times 5 = 15$, or $35$, in the box which is recorded as $(20 + 15) = 35$.

Suppose we have 4 of these boxes, there are two methods to find their total:

I. Each box has $35$ so 4 boxes have $4 \times 35 = 140$ in total.
II. Unpack the boxes, there are $4 \times 20 = 80$ from the $10$-bills, and $4 \times 15 = 60$ from the $5$-bills, which total to $140$.

Recording these with symbols:

$4(20+15) = 4(35)$

or $4(20+15) = 4(20) + 4(15) = 140$

The point here is that these two ways of totaling always yield the same result, not which is easier.

The fact that these two ways are identical and that there two forms associated with these two ways is important. Let's redo this in symbols.

The Distributive Law

There are two $10$-bills and three $5$-bills in a box.

Using the letter $T$ to represent a $10$-bill and $F$ for a $5$-bill, we may represent this box as $(2T + 3F)$.

If we have four of these boxes, we may record them as $4(2T + 3F)$. We open all the boxes and take out all the bills, there are 4 sets of $2T$'s or $8T$'s and 4 sets of $3F$'s or $12F$'s.

We track this unpacking or expanding procedure, as

$$4(2T + 3F) = 4 \times 2T + 4 \times 3F = 8T + 12F$$

**The factored form.**  **The expanded form.**

Expressing this operation in symbol we have the

**Distributive Law:**

$$a \times (b + c) = a \times b + a \times c$$

The "packed" form is **the factored form.**  The "unpacked" form is **the expanded form.**
Example E. A fruit basket contains 4 apples and 6 bananas, using A for "an apple" and B for "a banana" write the expression for a fruit basket, then write the expression for 5 such baskets. Expand the expression, how many apples and bananas do we have?

The basket may be written as (4A + 6B).
Five baskets are 5(4A + 6B), distribute the 5 and expand,
5(4A + 6B) = 5·4A + 5·6B = 20A + 30B
so that we have 20 apples and 30 bananas.

B. We unpack six fruit boxes and obtain 48 apples and 30 bananas. Record this amount using A and B (i.e. the expanded form), then write it in the factored form (i.e. as 6 packed boxes). What were in each box?
The fruits may be recorded as 48A + 30B, packed in six baskets, so each has 48/6 = 8 apples and 30/6 = 5 bananas, hence 48A + 30B = 6(8A + 5B) in the factored form.

The Basics

F1–Exercise A.
Do the following problems two ways.
* Add the following by summing the multiples of 10 first.
* Add by adding in the order.
to find the correct answer.

1. 3 + 5 + 7
2. 8 + 6 + 2
3. 1 + 8 + 9
4. 3 + 5 + 15
5. 9 + 14 + 6
6. 22 + 5 + 8
7. 16 + 5 + 4 + 3
8. 4 + 13 + 5 + 7
9. 19 + 7 + 1 + 3
10. 4 + 5 + 17 + 3
11. 23 + 5 + 17 + 3
12. 22 + 5 + 13 + 28
13. 35 + 6 + 15 + 7 + 14
14. 42 + 5 + 18 + 12
15. 21 + 16 + 19 + 7 + 44
16. 53 + 5 + 18 + 27 + 22
17. 155 + 16 + 25 + 7 + 344
18. 428 + 3 + 32 + 227 + 22
The Basics

1. $3^3 \quad 2. 4^2 \quad 3. 5^2 \quad 4. 5^3 \quad 5. 2^6 \quad 6. 6^3 \quad 7. 7^2
2. $8^2 \quad 9. 9^2 \quad 10. 10^2 \quad 11. 10^3 \quad 12. 10^4 \quad 13. 10^5
3. $14. 100^2 \quad 15. 100^3 \quad 16. 100^4 \quad 17. 11^2 \quad 18. 12^2
19. List the all the factors and the first 4 multiples of the following numbers. 6, 9, 10, 15, 16, 24, 30, 36, 42, 56, 60.
20. Factor completely and arrange the factors from smallest to the largest in the exponential notation: 4, 8, 12, 16, 18, 24, 27, 32, 36, 45, 48, 56, 60, 63, 72, 75, 81, 120.

C. Multiply in two ways to find the correct answer.
21. $3 \cdot 5 \cdot 4 \cdot 2 \quad 22. 6 \cdot 5 \cdot 4 \cdot 3 \quad 23. 6 \cdot 15 \cdot 3 \cdot 2$
24. $7 \cdot 5 \cdot 4 \quad 25. 6 \cdot 7 \cdot 4 \cdot 3 \quad 26. 9 \cdot 3 \cdot 4 \cdot 4$
27. $2 \cdot 25 \cdot 3 \cdot 4 \cdot 2 \quad 28. 3 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 4$
29. $3 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 4 \quad 30. 4 \cdot 2 \cdot 3 \cdot 15 \cdot 8 \cdot 4$
31. $2^4 \quad 32. 2^5 \quad 33. 2^6 \quad 34. 2^7 \quad 35. 2^8$
36. $2^9 \quad 37. 2^{10} \quad 38. 3^4 \quad 39. 3^5 \quad 40. 3^6$

41. a. A fruit basket contains 7 apples and 5 bananas, using A for “an apple” and B for “a banana” write the expression for a fruit basket,
b. Write the expression for 4 such baskets.
c. Expand the expression, how many apples and bananas do we have?
42. We unpack four fruit boxes and obtain 20 apples and 32 bananas.
a. Express this using A and B in the expanded form.
b. Express this in the factored form, i.e. as 4 packed boxes. What were in each box?
Fractions are numbers of the form $\frac{p}{q}$ (or $p/q$) where $p$, $q$ are natural numbers: 1, 2, 3,...etc. Fractions are numbers that measure parts of whole items. Suppose a pizza is cut into 6 equal slices and we have 3 of them, the fraction that represents this quantity is $\frac{3}{6}$.

The top number “3” is the number of parts that we have and it is called the numerator.

The bottom number is the number of equal parts in the division and it is called the denominator.

For larger denominators we can use a pan–pizza for pictures. For example.

$\frac{5}{8}$

$\frac{7}{12}$

Note that $\frac{8}{8}$ or $\frac{12}{12}$ is the same as 1.

Fact: $\frac{a}{a} = 1$ (provided that $a \neq 0$.)
Whole numbers can be viewed as fractions with denominator 1. Thus \(5 = \frac{5}{1}\) and \(x = \frac{x}{1}\). The fraction \(\frac{0}{x}\) = 0, where \(x \neq 0\).

However, \(\frac{x}{0}\) does not have any meaning. It is undefined.

The Ultimate No-No of Mathematics:
The denominator (bottom) of a fraction can’t be 0. (It’s undefined if the denominator is 0.)

Fractions that represent the same quantity are called equivalent fractions.

\[
\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} \ldots \text{ are equivalent fractions.}
\]

The fraction with the smallest denominator of all the equivalent fractions is called the reduced fraction,

\(\frac{1}{2}\) is the reduced one in the above list.

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Fractions

Factor Cancellation Rule

For any fraction \(\frac{x}{y}\), \(\frac{x}{y} = \frac{x/c}{y/c}\) where \(c \neq 0\),

that is, if the numerator and denominator are divided by the same quantity \(c\), the result will be an equivalent fraction.

To reduce a fraction, we keep divide the top and bottom by common numbers until no more division is possible.

What's left is the reduced version.

Example A. Reduce the fraction.

a. \(\frac{9}{15} = \frac{9/3}{15/3} = \frac{3}{5}\)

b. \(\frac{78}{54} = \frac{78/2}{54/2} = \frac{39/3}{27/3} = \frac{13}{9}\)

(or divide by 6 directly)

Hence a common factor of the numerator and the denominator may be canceled as 1, so \(\frac{a \cdot c}{b \cdot c} = \frac{a/c}{b/c} = \frac{a}{b}\)

(Often we omit writing the 1’s after the cancellation.)
Fractions

Example B. A pizza shop sells pizza by the slices. Each slice is 1/12th of a pizza. Different customers order different number of slices. Find the least number of slices they could cut the pizza into and still fill the following orders and how many of the newly cut slices each order needs? Draw.

a. Joe wants 8 slices.
   Joe wants $\frac{8}{12}$. We note that both 8 and 12 may be grouped into 4s as shown. Hence $\frac{8\div4}{12\div4} = \frac{2}{3}$ or that Joe gets 2 slices out of a pizza cut into 3 slices.

b. Mary wants 10 slices.
   Both 10 and 12 are divisible by 2.
   $\frac{10}{12} = \frac{5}{6}$ and Mary gets 5 out of the 6 slices.

Fractions

One common mistake in cancellation is to cancel a common number that is part of an addition (or subtraction) in the numerator or denominator.

A participant in a sum or a difference is called a term.

The “2” in the expression “2 + 3” is a term (of the expression). The “2” is in the expression “2 * 3” is called a factor.

Terms may not be cancelled. Only factors may be canceled.

\[
\frac{3}{5} = \frac{2 + 1}{2 + 3} \neq \frac{2 + 1}{2 + 3} = \frac{3}{5}
\]

This is addition. Can’t cancel!

Improper Fractions and Mixed Numbers

A fraction whose numerator is the same or more than its denominator (e.g. $\frac{3}{2}$) is said to be improper.

We may put an improper fraction into mixed form by division.
Improper Fractions and Mixed Numbers

Exercise. A. Reduce the following fractions.
\[
\frac{4}{6}, \frac{8}{12}, \frac{15}{18}, \frac{24}{42}, \frac{30}{36}, \frac{54}{48}, \frac{60}{108}
\]

B. Convert the following improper fractions into mixed numbers then convert the mixed numbers back to the improper form.

1. \(\frac{9}{2}\)  2. \(\frac{11}{3}\)  3. \(\frac{9}{4}\)  4. \(\frac{13}{5}\)  5. \(\frac{37}{12}\)  6. \(\frac{86}{11}\)  7. \(\frac{121}{17}\)
1–f3 Multiplication and Division of Fractions

Fractions \( \frac{a}{b} \) (or \( p/q \)) are numbers that measure parts of whole items. For example, \( \frac{3}{6} \) of a pizza represents “3 out of 6 slices” as shown here.

The top number “3” is the number of parts that we have and it is called the numerator.

The bottom number is the number of equal parts in the division and it is called the denominator.

Example A. Joe took 2/3 of a dozen eggs, how many eggs did Joe take?
The fraction 2/3 means to divide the dozen eggs into 3 piles, so each pile has 12/3 = 4 eggs. Joe took two piles so he took 2 \times 4 = 8 eggs. The process is recorded as multiplication:

\[
\frac{2}{3} \cdot 12 = \frac{2}{3} \cdot \frac{12}{1} = 2 \cdot 4 = 8
\]

Step 1. 12 \div 3 = 4
so each pile has 4 eggs.

Step 2. 2 \times 4 = 8 eggs
so 2 piles has 8 eggs.

Multiplication and Division of Fractions

The fractional portion \( \frac{a}{b} \) of a whole number \( x \) is expressed as \( \frac{a}{b} \cdot \frac{x}{1} \) or \( \frac{x}{1} \cdot \frac{a}{b} \). To simplify these, always divide \( x \) by \( b \) or cancel the common factor of \( x \) and \( b \) first.

Example B. Multiply by cancelling first.

\[
a. \quad \frac{2}{3} \cdot \frac{18}{6} = \frac{2 \cdot 6}{3 \cdot 3} = \frac{12}{9} = \frac{4}{3} \quad \text{do } 18 \div 3 = 6 \text{ first}
\]

\[
b. \quad \frac{3}{16} \cdot \frac{11}{1} = \frac{3 \cdot 11}{16} = \frac{33}{16} \quad \text{do } 48 \div 16 = 3 \text{ first}
\]

The often used phrases "(fraction) of .." are translated to multiplications correspond to this kind of problems.

Example C. a. What is \( \frac{2}{3} \) of $108?
The statement translates into \( \frac{2}{3} \cdot 108 = \frac{2 \cdot 36}{3} = 2 \cdot 36 = 72 \$.
So 2/3 of $108 is $72. (do 108/3 = 36 first)
Multiplication and Division of Fractions

b. A bag of mixed candy contains 48 pieces of chocolate, caramel and lemon drops, 1/4 of them are chocolate, 1/3 of them are caramel. How many pieces of each are there? What fraction of the candies are lemon drops?

For chocolate, 1/4 of 48 is \( \frac{1}{4} \cdot 48 = \frac{48}{4} = 12 \),
so there are 12 pieces of chocolate candies.

For caramel, 1/3 of 48 is \( \frac{1}{3} \cdot 48 = \frac{48}{3} = 16 \),
so there are 16 pieces of caramel candies.
The rest 48 – 12 – 16 = 20 are lemon drops.

So the fraction of the lemon drops is \( \frac{20}{48} = \frac{20/4}{48/4} = \frac{5}{12} \).

Example C. What is 1/4 of 10 lb flour?

"1/4 of 10" is \( \frac{1}{4} \cdot 10^5 \) (Reduce 10 and 4 by dividing by 2)

\[ = \frac{1}{2} \cdot \frac{5}{2} = \frac{5}{4} \] or \( 2 \frac{1}{2} \) lb of flour.

---

**Multiplication of Two Fractions**

In general, we multiply two fractions as shown,

\[ \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \]

but as in the above examples, the keys is to reduce the product by canceling any common "top-and-bottom factor", as much as possible first, then multiply.

Example B. Multiply by reducing first.

a. \( \frac{15}{8} \cdot \frac{12}{25} = \frac{3 \cdot 15 \cdot 12}{2 \cdot 25} = \frac{3 \cdot 3}{2 \cdot 5} = \frac{9}{10} \)

b. \( \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10} = \frac{7 \cdot 8 \cdot 9}{8 \cdot 9 \cdot 10} = \frac{7}{10} \)

*Can't do this for addition and subtraction, i.e. \( \frac{a}{b} \pm \frac{c}{d} \neq \frac{a \pm c}{b \pm d} \).*
Reciprocal and Division of Fractions

The reciprocal (multiplicative inverse) of \( \frac{a}{b} \) is \( \frac{b}{a} \).

So the reciprocal of \( \frac{2}{3} \) is \( \frac{3}{2} \), the reciprocal of 5 is \( \frac{1}{5} \).

the reciprocal of \( \frac{1}{3} \) is 3, and the reciprocal of \( x \) is \( \frac{1}{x} \).

**Two Important Facts About Reciprocals**

I. The product of \( x \) with its reciprocal is 1.

\[
\frac{2}{3} \cdot \frac{3}{2} = 1, \quad 5 \cdot \frac{1}{5} = 1, \quad x \cdot \frac{1}{x} = 1,
\]

II. Dividing by \( x \) is the same as multiplying by its reciprocal \( \frac{1}{x} \).

For example, \( 10 \div 2 \) is the same as \( 10 \cdot \frac{1}{2} \), both yield 5.

**Rule for Division of Fractions**

To divide by a fraction \( x \), restate it as multiplying by the reciprocal \( \frac{1}{x} \), that is,

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}
\]

Example F. Divide the following fractions.

a. \( \frac{12}{25} \div \frac{8}{15} = \frac{12}{25} \cdot \frac{15}{8} = \frac{9}{10} \)

b. \( \frac{9}{8} \div 6 = \frac{9}{8} \cdot \frac{1}{6} = \frac{3}{16} \)

d. \( 5 \div \frac{1}{6} = 5 \cdot \frac{6}{1} = 30 \)

Example F. We have \( \frac{3}{4} \) cups of sugar. A cookie recipe calls for \( \frac{1}{16} \) cup of sugar for each cookie. How many cookies can we make?

We can make \( \frac{3}{4} \div \frac{1}{16} = \frac{3}{4} \cdot \frac{16}{1} = 3 \cdot 4 = 12 \) cookies.

Hence, all division problems may be treated as multiplication.

HW: Do the web homework "Multiplication of Fractions"
Multiplication and Division of Fractions

Exercise: Do the following problems like the example above. Do them mentally.

1. a. \(\frac{3}{2} \times 4\) b. \(\frac{5}{2} \times 4\) c. \(\frac{7}{2} \times 4\) d. \(\frac{9}{2} \times 4\)
2. a. \(\frac{3}{2} \times 6\) b. \(\frac{2}{3} \times 6\) c. \(\frac{5}{2} \times 6\) d. \(\frac{4}{3} \times 6\)
3. a. \(\frac{5}{3} \times 6\) b. \(\frac{5}{2} \times 6\) c. \(\frac{5}{6} \times 6\) d. \(\frac{9}{2} \times 6\)
4. a. \(\frac{3}{4} \times 8\) b. \(\frac{5}{2} \times 8\) c. \(\frac{7}{4} \times 8\) d. \(\frac{5}{4} \times 8\)
5. a. \(\frac{3}{9} \times 9\) b. \(\frac{5}{3} \times 9\) c. \(\frac{5}{4} \times 12\) d. \(\frac{9}{2} \times 12\)
6. a. \(\frac{3}{12} \times 12\) b. \(\frac{5}{2} \times 12\) c. \(\frac{7}{6} \times 12\) d. \(\frac{4}{3} \times 12\)

Remember to cancel first!

Multiplication and Division of Fractions

7. a. \(\frac{3}{2} \times 18\) b. \(\frac{4}{3} \times 18\) c. \(\frac{7}{6} \times 18\) d. \(\frac{7}{9} \times 18\)
8. a. \(\frac{3}{2} \times 24\) b. \(\frac{5}{3} \times 24\) c. \(\frac{7}{4} \times 24\) d. \(\frac{5}{6} \times 24\)
9. a. \(\frac{7}{12} \times 24\) b. \(\frac{11}{6} \times 24\) c. \(\frac{7}{4} \times 36\) d. \(\frac{5}{6} \times 36\)
10. a. \(\frac{3}{2} \times 36\) b. \(\frac{5}{9} \times 36\) c. \(\frac{7}{12} \times 36\) d. \(\frac{5}{3} \times 36\)

11. In a class of 45 people, \(\frac{2}{3}\) are boys, how many boys are there?

12. Factor completely and write the answer using the exponential notation

36, 56, 60, 75, 108, 180, 360
Multiplication and Division of Fractions

13. A bag has 144 jelly beans, \(\frac{r}{9}\) are red, \(\frac{5}{8}\) are blue, the rest are green. How many of each type are there?

14. In room of 120 people, \(\frac{1}{4}\) are male children, \(\frac{3}{8}\) are female adults. There are 83 adults. Complete the table.

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adults</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Children</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

15. How many sticks with length of \(1\frac{3}{8}\) ft each are needed to make a total length of 22 feet.
1–f4 LCM and LCD

Example A.
Sam has two bags of apples both with the same number of apples. He gave one bag to Abe, Bolo and Cato and they share the bag evenly—no left over, he gave the other bag to Dora, Eva, Fifi and Goomba and they are able to share the bag evenly also. What is the least number of apples possible in each bags?

The answer is 12.
The number of apples must be multiples of 3: 3, 6, 9, 12, 15,
to be shared evenly by 3 people: Abe, Bolo and Cato.
It must be multiples of 4: 4, 8, 12, 16,...
to be shared by 4 people: Dora, Eva, Fifi and Goomba.
The least multiple on both lists is 12.
Hence the least number of apples in the bag is 12.
12 is known as the least common multiple (LCM) of 3 and 4.
Note that the actual number of apples could also be multiples of 12 such as 24, 36,... etc.

LCM and LCD

Definition of LCM
The *least common multiple (LCM)* of a group of two or more numbers is the smallest number that is the multiple of all of these numbers.

To find the LCM of a few small numbers,
list the multiples of each number orders until the we find first common one which would also be the least one.

Example B. Find the LCM of 8, 18, and 12.
We may shorten the work by only listing the multiples of the largest number in the group, which is 18,
until we find the one that can divide by 8 and also by 12.
The multiples of 18 are 18, 36, 54, 72, 90,...
The first number that is also a multiple of 8 and 12 is 72.
Hence LCM{8, 18, 12} = 72.
LCM and LCD

But when the LCM is large, the listing method is cumbersome. It's easier to find the LCM by constructing it instead.

To construct the LCM:

a. Factor each number completely
b. For each prime factor, take the highest power appearing in the factorizations. The LCM is their product.

Example C. Construct the LCM of \{8, 15, 18\}.

Factor each number completely,

\[
\begin{align*}
8 &= 2^3 \\
15 &= 3 \cdot 5 \\
18 &= 2 \cdot 3^2
\end{align*}
\]

From the factorization select the highest degree of each prime factor: \(2^3, 3^2, 5\), then \(\text{LCM}(8, 15, 18) = 2^3 \cdot 3^2 \cdot 5 = 8 \cdot 9 \cdot 5 = 360\).

The LCM of the denominators of a list of fractions is called the least common denominator (LCD).

---

Example D. From one pizza, Joe wants \(\frac{1}{3}\), Mary wants \(\frac{1}{4}\) and Chuck wants \(\frac{1}{6}\). How many equal slices should we cut the pizza into and how many slices should each person take? What is the fractional amount of the pizza they want in total?

In picture:

We find the LCM of \(\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\) by searching. The multiples of 6 are 6, 12, 18, 24, ... Since 12 is also the multiple of 3 and 4, then 12 is the LCM. Hence we should cut it into 12 slices and

Joe gets \(12 \cdot \frac{1}{3} = 4\) slices

Mary gets \(12 \cdot \frac{1}{4} = 3\) slices

Chuck gets \(12 \cdot \frac{1}{6} = 2\) slices

In total, that is \(4 + 2 + 3 = 9\) slices, or \(\frac{9}{12} = \frac{3}{4}\) of the pizza.
LCM and LCD

Your Turn: Joe wants $\frac{3}{8}$, Mary wants $\frac{1}{6}$ and Chuck wants $\frac{5}{12}$ of a pizza. How many equal slices should we cut the pizza into and how many slices should each person get?

Recall that $\frac{2}{3}$ of 12 eggs are $\frac{2}{3} \cdot 12 = 8$ eggs.

Another form of this question is to find $x$ if $\frac{2}{3} = \frac{x}{12}$, and we see that $x = \frac{2}{3} \cdot 12 = 8$ eggs.

Expand the Denominator of a Fraction

To expand the fraction $\frac{a}{b}$ to $\frac{x}{d}$, with a new denominator $d$, the new numerator $x = \frac{a}{b} \cdot d$.

Example E. Convert $\frac{9}{16}$ to a fraction with denominator 48.

The new denominator is 48, so the new numerator is $\frac{3 \cdot 9}{48} = 27$ so $\frac{9}{16} = \frac{27}{48}$.

LCM and LCD

Here is another example of fulfilling "the minimal" requirements.

Example F. There are three identical boxes with the same content. Apu took some items from the 1st box and what's left is shown here. Bolo took some items from the 2nd box and what's left is shown here. Cato took some items from the 3rd box and what's left is shown here.

What is the least amount of items possible in box originally?
What's the least amount of items that each person took?

The least amount of items in the box consist of 2 apples, 5 bananas and 4 carrots. Apu took 1 apple and 1 banana, Bolo took 2 carrots, and Cato took 2 banana and 1 carrot.
LCM and LCD

Exercise A. Find the LCM.
1. a. \(\{6, 8\}\)  b. \(\{6, 9\}\)  c. \(\{3, 4\}\)
   d. \(\{4, 10\}\)
2. a. \(\{5, 6, 8\}\)  b. \(\{4, 6, 9\}\)  c. \(\{3, 4, 5\}\)
   d. \(\{4, 6, 10\}\)
3. a. \(\{6, 8, 9\}\)  b. \(\{6, 9, 10\}\)  c. \(\{4, 9, 10\}\)
   d. \(\{6, 8, 10\}\)
4. a. \(\{4, 8, 15\}\)  b. \(\{8, 9, 12\}\)  c. \(\{6, 9, 15\}\)
5. a. \(\{6, 8, 15\}\)  b. \(\{8, 9, 15\}\)  c. \(\{6, 9, 16\}\)
6. a. \(\{8, 12, 15\}\)  b. \(\{9, 12, 15\}\)  c. \(\{9, 12, 16\}\)
7. a. \(\{8, 12, 18\}\)  b. \(\{8, 12, 20\}\)  c. \(\{12, 15, 16\}\)
8. a. \(\{8, 12, 15, 18\}\)  b. \(\{8, 12, 16, 20\}\)
9. a. \(\{8, 15, 18, 20\}\)  b. \(\{9, 16, 20, 24\}\)


LCM and LCD

B. Convert the fractions to fractions with the given denominators.
10. Convert \(\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{4}\) to denominator 12.
11. Convert \(\frac{1}{6}, \frac{3}{4}, \frac{5}{6}, \frac{3}{8}\) to denominator 24.
12. Convert \(\frac{7}{12}, \frac{5}{4}, \frac{8}{9}, \frac{11}{6}\) to denominator 36.
13. Convert \(\frac{9}{10}, \frac{7}{12}, \frac{13}{5}, \frac{11}{15}\) to denominator 60.
14. Convert \(\frac{5}{8}, \frac{7}{12}, \frac{13}{15}, \frac{17}{24}\) to denominator 120.
1–f5 Addition and Subtraction of Fractions

Suppose a pizza is cut into 4 equal slices and Joe takes one slice or $\frac{1}{4}$ of the pizza, Mary takes two slices or $\frac{2}{4}$ of the pizza, altogether they take

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

of the entire pizza. In picture:

Addition and Subtraction of Fractions With the Same Denominator

To add or subtract fractions of the same denominator, keep the same denominator, add or subtract the numerators

$$\frac{a}{d} \pm \frac{b}{d} = \frac{a \pm b}{d}$$

then simplify the result.

Addition and Subtraction of Fractions

Example A.

a. $\frac{7}{12} + \frac{11}{12} = \frac{7 + 11}{12} = \frac{18}{12} = \frac{18 \div 6}{12 \div 6} = \frac{3}{2}$

b. $\frac{8}{15} + \frac{4}{15} - \frac{2}{15} = \frac{8 + 4 - 2}{15} = \frac{10}{15} = \frac{2}{3}$

Subtraction of Whole Numbers with Fractions

Example B. a. Bolo ate $\frac{5}{8}$ of 1 pizza, what’s left?

Treating 1 as $\frac{8}{8}$, what’s left is: $1 - \frac{5}{8} = \frac{8}{8} - \frac{5}{8} = \frac{3}{8}$

b. There were 3 pizzas on the table and Bolo ate $\frac{7}{9}$ of 1 pizza, what’s left?

Of the 3 pizzas, $\frac{2}{9}$ is left of the eaten one, and 2 pizzas are untouched. So $3 - \frac{7}{9} = 1 - \frac{7}{9} + 2 = 2 \frac{2}{9}$ pizzas are left.
Addition and Subtraction of Fractions

c. There were 8 pizzas on the table and Bolo ate \(3\frac{2}{5}\) pizzas, how much pizzas are left?

Of the 8 pizzas, 3 are eaten with 5 left, then another \(\frac{2}{5}\) of one is eaten. So \(8 - 3\frac{2}{5} = 8 - 3 - \frac{2}{5} = 5 - \frac{2}{5} = 4\frac{3}{5}\) are left.

To add/subtract (±) mixed fractions of the same denominator, (±) the whole number parts first, then (±) the fractional parts of which it might be necessary to carry or borrow.

Example C. Calculate.

a. \(4\frac{5}{6} + 8\frac{7}{8} = 4 + 8 + \frac{5}{6} + \frac{7}{8}\)

\[= 12 + \frac{12}{8}\]
\[= 12 + 1\frac{1}{2}\]
\[= 13\frac{1}{2}\]

Addition and Subtraction of Fractions

c. \(8\frac{2}{5} - 3\frac{4}{5}\)

\[8\frac{2}{5} - 3\frac{4}{5} = 8\frac{2}{5} - 3 - \frac{4}{5} = 5\frac{2}{5} - \frac{4}{5}\]

Borrow 1 to subtract 4/5

\[= 4 + 1\frac{2}{5} - \frac{4}{5} = 4 + \frac{7}{5} - \frac{4}{5} = 4\frac{3}{5}\]

Fractions with different denominators can’t be added directly since the “size” of the fractions don’t match. For example

\[\text{To add them, first find the LCD of } \frac{3}{8} \text{ and } \frac{2}{6}, \text{ which is 6. We then cut each pizza into 6 slices. Both fractions may be converted to have the denominator 6. Specifically,}\]

\[\frac{1}{2} = \frac{3}{6}, \quad \frac{1}{3} = \frac{2}{6} \quad \text{Hence,} \quad \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}\]
Addition and Subtraction of Fractions

We need to convert fractions of different denominators to a common denominator in order to add or subtract them. The easiest common denominator to use is the LCD, the least common denominator. We list the steps below.

Addition and Subtraction of Fractions With the Different Denominator (The Traditional Method)

1. Find their LCD
2. Convert all the different-denominator-fractions to the have the LCD as the denominator.
3. Add and subtract the adjusted fractions then simplify the result.

Example D. a. $\frac{5}{6} + \frac{3}{8}$

Step 1: To find the LCD, list the multiples of 8 which are 8, 16, 24, .. we see that the LCD is 24.

Addition and Subtraction of Fractions

Step 2: Convert each fraction to have 24 as the denominator.

For $\frac{5}{6}$, the new numerator is $24 \cdot \frac{5}{6} = 20$,
so $\frac{5}{6} = \frac{20}{24}$

For $\frac{3}{8}$, the new numerator is $24 \cdot \frac{3}{8} = 9$,
so $\frac{3}{8} = \frac{9}{24}$

Step 3: Add the converted fractions.

So $\frac{5}{6} + \frac{3}{8}$

$= \frac{20}{24} + \frac{9}{24}$

$= \frac{29}{24}$ (It’s reduced.)
Addition and Subtraction of Fractions

We introduce the following Multiplier-Method to add or subtract fractions to reduce the amount of repetitive copying. This method is based on the fact that if we multiply a quantity x by a, then divide by a, we get back x. For example, \(2 \cdot \frac{5}{5} = \frac{10}{5} = 2\), \(3 \cdot \frac{8}{8} = \frac{24}{8} = 3\).

**Multiplier Method** (for adding and subtracting fractions)

To add or subtract fractions, multiply the problem by the LCD, distributive to find the numerator over the LCD for the answer.

Example E. a. \(\frac{5}{6} + \frac{3}{8}\)

The LCD is 24. Multiply the problem by 24, then divide by 24.

\[
\left( \frac{4 \cdot 5}{24} + \frac{3 \cdot 3}{24} \right) \cdot \frac{24}{24} \quad \text{distribute the multiplication}
\]

\[
= \left( \frac{20}{24} + \frac{9}{24} \right) / 24
\]

\[
= \frac{29}{24}
\]

Addition and Subtraction of Fractions

b. \(\frac{7}{12} + \frac{5}{8} - \frac{9}{16}\)

The LCD is 48. Multiply the problem by 48, expand the multiplication, then divide the result by 48.

\[
\left( \frac{4 \cdot 7}{48} + \frac{6 \cdot 5}{48} - \frac{3 \cdot 9}{48} \right) \cdot \frac{48}{48}
\]

\[
= \left( \frac{28}{48} + \frac{30}{48} - \frac{27}{48} \right) / 48
\]

\[
= \frac{31}{48}
\]

The Multiplier-Method would be the main method used for adding/subtracting fractional numbers and formulas throughout this and following courses.
Addition and Subtraction of Fractions

Exercise A. Calculate and simplify the answers.

1. $\frac{1}{2} + \frac{3}{2}$  
2. $\frac{5}{3} + \frac{1}{3}$  
3. $\frac{5}{4} + \frac{3}{4}$  
4. $\frac{5}{2} + \frac{3}{2}$  
5. $\frac{5}{5} - \frac{3}{5}$  
6. $\frac{6}{6} - \frac{5}{6}$  
7. $\frac{9}{9} - \frac{4}{9}$  
8. $1 - \frac{4}{7}$  
9. $1 - \frac{2}{9}$  
10. $1 - \frac{3}{8}$  
11. $4 - \frac{3}{4}$  
12. $8 - \frac{3}{8}$  
13. $11 - \frac{3}{5}$  
14. $\frac{9}{8} - 5 - \frac{3}{8}$  
15. $14 - 6 - \frac{1}{6}$  
16. $21 - \frac{5}{11} - 8 - \frac{9}{11}$  

B. Calculate by the Multiplier Method and simplify the answers.

17. $\frac{1}{2} + \frac{1}{3}$  
18. $\frac{1}{2} - \frac{1}{3}$  
19. $\frac{2}{3} + \frac{3}{2}$  
20. $\frac{3}{4} + \frac{2}{5}$  
21. $\frac{5}{6} - \frac{4}{7}$  
22. $\frac{7}{10} - \frac{2}{5}$  
23. $\frac{5}{11} + \frac{3}{4}$  
24. $\frac{5}{9} - \frac{7}{15}$  

C. Addition and Subtraction of Fractions

25. $\frac{5}{4} + \frac{1}{6}$  
26. $\frac{7}{4} - \frac{5}{6}$  
27. $\frac{5}{9} + \frac{7}{12}$  
28. $\frac{3}{8} + \frac{5}{12}$  
29. $\frac{7}{24} + \frac{5}{16}$  
30. $\frac{7}{12} - \frac{5}{18}$  
31. $\frac{11}{24} - \frac{3}{20}$  
32. $\frac{5}{18} + \frac{7}{15}$  
33. $\frac{3}{4} + \frac{1}{6} - \frac{4}{9}$  
34. $\frac{5}{4} - \frac{1}{6} - \frac{7}{10}$  
35. $\frac{3}{8} - \frac{1}{6} - \frac{5}{12}$  
36. $\frac{7}{8} - \frac{5}{9} - \frac{1}{12}$  
37. $\frac{5}{24} + \frac{1}{16} - \frac{2}{9}$  
38. $\frac{5}{4} - \frac{1}{12} - \frac{7}{18}$  
39. $\frac{5}{18} - \frac{1}{16} - \frac{7}{12}$  
40. $\frac{7}{24} - \frac{5}{18} + \frac{7}{10}$
Addition and Subtraction of Fractions

41. $\frac{5}{4} - \frac{5}{6} + \frac{5}{12}$
42. $\frac{3}{8} + \frac{4}{9} - \frac{5}{6}$

43. $\frac{7}{18} + \frac{4}{15} - \frac{7}{12}$
44. $\frac{19}{24} - \frac{7}{15} + \frac{11}{18}$

45. $5 - \frac{7}{12} + \frac{5}{8}$
46. $3 \frac{5}{8} - \frac{7}{9} - \frac{1}{6}$
### 1–f6 Cross Multiplication

The importance of the LCD is that the LCD is the smallest, hence the easiest number to “multiply” all the fractions in question into whole numbers.

**Example A.**

a. Find the LCD of the following list of fractions, \{2/3, 5/8, 7/12, 3/4\}.

List the multiples of 12 (from 7/12): 12, 24, 36, . . .

We find that the LCD is 24.

b. Multiply the LCD to each fraction to change the list into a whole-number list.

\[
\begin{align*}
\{2/3, 5/8, 7/12, 3/4\} & \times 24 \\
\{8/24, 3/24, 14/24, 18/24\} & \\
\{2/3, 5/8, 7/12, 3/4\} & \times 24
\end{align*}
\]

c. List the fractions from the largest to the smallest.

From the whole numbers, listing the fractions from the largest to the smallest:

\[
\{16, 15, 14, 13\}
\]

\[
\begin{align*}
3/4, & \quad 2/3, & \quad 5/8, & \quad 7/12
\end{align*}
\]

### Cross Multiplication

With two fractions, here is an easier and useful procedure for clearing their denominators.

**Cross Multiplication**

Given two fractions as shown below, multiplying to each fraction by the common denominator bd (which may not be the LCD), to clear the denominators, yield the same outcome as:

*taking the denominators and multiply them diagonally across.*

\[
\begin{align*}
\frac{ad}{bc} & \quad \text{(cross-multiplication)}
\end{align*}
\]

Be sure the denominators *cross over and up*
so the outcomes correspond to the fractions.

**Do not cross downward as shown here!!**

(The results are out of order!)
Cross Multiplication

Here are some operations where we may cross multiply.

Rephrasing Fractional Ratios

If a cookie recipe calls for 3 cups of sugar and 2 cups of flour, we say the ratio of sugar to flour is 3 to 2, and it’s written as 3 : 2 for sugar : flour. Or we said the ratio of flour : sugar is 2 : 3. For most people, a recipe that calls for the fractional ratio of 3/4 cup sugar to 2/3 cup of flour is confusing. It’s better to cross multiply to rewrite this ratio in whole numbers.

Example B.

rewrite a recipe that calls for the fractional ratio of 3/4 cup sugar to 2/3 cup of flour into ratio of whole numbers.

Write 3/4 cup of sugar as \( \frac{3}{4}S \) and 2/3 cup of flour as \( \frac{2}{3}F \).

We have the ratio \( \frac{3}{4}S : \frac{2}{3}F \) cross multiply we’ve 9S : 8F.

Hence in integers, the ratio is 9 : 8 for sugar : flour.

Remark: A ratio such as 8 : 4 should be simplified to 2 : 1.

Cross Multiplication

Cross–Multiplication Test for Comparing Two Fractions

When comparing two fractions to see which is larger and which is smaller. Cross–multiply them, the side with the larger product corresponds to the larger fraction.

In particular, if the cross multiplication products are the same then the fraction are the same.

Hence cross– multiply \( \frac{3}{5} \times \frac{9}{15} \) we get

\[
\frac{3}{5} \times \frac{9}{15} = \frac{45}{75} = \frac{9}{15}
\]

so

\[
\frac{3}{5} < \frac{9}{15}
\]

Cross– multiply \( \frac{3}{5} \times \frac{5}{8} \) we get

\[
\frac{3}{5} \times \frac{5}{8} = \frac{15}{40} = \frac{24}{40}
\]

less

\[
\frac{3}{5} < \frac{5}{8}
\]

Hence \( \frac{3}{5} \) is less than \( \frac{5}{8} \).

(Which is more \( \frac{9}{14} \) or \( \frac{7}{11} \)? Do it by inspection.)
Cross Multiplication

Cross–Multiplication for Addition or Subtraction
We may cross multiply to add or subtract two fractions with the product of the denominators as the common denominator.

\[ \frac{a}{b} \times \frac{c}{d} = \frac{ad + bc}{bd} \]

Afterwards we reduce if necessary for the simplified answer.

Example C. Calculate
a. \( \frac{5}{6} \times \frac{3}{5} = \frac{5 \times 5 - 6 \times 3}{6 \times 5} = \frac{7}{30} \)

b. \( \frac{5}{9} \times \frac{5}{12} = \frac{5 \times 12 - 9 \times 5}{9 \times 12} = \frac{15}{108} = \frac{5}{36} \)

In a. the LCD = 30 = 6*5 so the crossing method is the same as the Multiplier Method. However in b. the crossing method yielded an answer that needed to be reduced. we need both methods.

Cross Multiplication

The Double Check Strategy
One of the most difficult thing to do in mathematics is to know that a mistake had taken place and to locate the mistake.

The Double Check is to cross check an answer by doing a problem two different ways. If both methods yielded the same answer then the answer is likely to be correct. If two answers are different then we have to clarify the mistake.

When we + or − fractions, we can use the above two methods to cross check an answer. For example, in part b. above, we obtain an answer via the crossing method. Let’s cross check the first answer using the Multiplier Method.

Since the LCD = 36, we multiply and divide by 36.

\[ \left( \frac{5}{12} - \frac{5}{12} \right) \times 36 / 36 \]
\[ = (5*4 - 5*3) / 36 = 5/36 \]

This is the same as before hence it’s very likely to be correct.
Cross Multiplication

Comments
* The Double Check Strategy is an important tool for learning. It reassures us if we’re heading in the right direction. It warns us that a mistake had occurred so we should back track and locate the mistake.

Use this Double Check Strategy for learning!
* The Multiplier Method and the Cross Multiplication Method are two methods to double check addition and subtraction of small number of fractions. These two methods generalize to addition and subtraction of fractional (rational) formulas in later topics. Each method leads to various ways of handling various fractional algebra problems where each way has its own advantages and disadvantage.

We use both methods throughout this database.

---

Cross Multiplication
Ex. Restate the following ratios in integers.

1. \( \frac{1}{2} : \frac{1}{3} \)
2. \( \frac{2}{3} : \frac{1}{2} \)
3. \( \frac{3}{4} : \frac{1}{3} \)
4. \( \frac{2}{3} : \frac{3}{4} \)
5. \( \frac{3}{5} : \frac{1}{2} \)
6. \( \frac{1}{6} : \frac{1}{7} \)
7. \( \frac{3}{5} : \frac{4}{7} \)
8. \( \frac{5}{2} : \frac{7}{4} \)

9. In a market, \( \frac{3}{4} \) of an apple may be traded with \( \frac{1}{2} \) a pear. Restate this using integers.

Determine which fraction is more and which is less.

10. \( \frac{2}{3} , \frac{3}{4} \)
11. \( \frac{4}{5} , \frac{3}{4} \)
12. \( \frac{4}{7} , \frac{3}{5} \)
13. \( \frac{5}{6} , \frac{4}{5} \)
14. \( \frac{5}{9} , \frac{4}{7} \)
15. \( \frac{7}{10} , \frac{2}{3} \)
16. \( \frac{5}{12} , \frac{3}{7} \)
17. \( \frac{13}{8} , \frac{8}{5} \)

C. Use cross–multiplication to combine the fractions.

18. \( \frac{1}{2} + \frac{1}{3} \)
19. \( \frac{1}{2} - \frac{1}{3} \)
20. \( \frac{2}{3} + \frac{3}{2} \)
21. \( \frac{3}{4} + \frac{2}{5} \)
22. \( \frac{5}{6} - \frac{4}{7} \)
23. \( \frac{7}{10} - \frac{2}{5} \)
24. \( \frac{5}{11} + \frac{3}{4} \)
25. \( \frac{5}{9} - \frac{7}{15} \)
1–f7 Some Facts About Divisibility
We start out with a simple mathematics procedure that is often used in real life. It’s called the digit sum. Just as its name suggests, we sum all the digits in a number. The digit sum of 12 is 1 + 2 = 3, is the same as the digit sum of 21, 111, or 11100. To find the digit sum of

\[
\begin{array}{c}
\text{add} \\
\hline
7 & 8 & 9 & 1 & 8 & 2 & 7 & 3 \\
\hline
15 & 30 & 45 & 9
\end{array}
\]

Hence the digit sum of 7899111 is 45.
If we keep adding the digits, the sums eventually become a single digit sum – the digit root.
The digit root of 78198273 is 9.

Some Facts About Divisibility
A bag contains 384 pieces of chocolate, can they be evenly divided by 12 kids? In mathematics, we ask “is 384 divisible by 12?”. How about 2,349,876,543,214 pieces of chocolate with 18 kids? Is 2,349,876,543,214 divisible by 18?
One simple application of the digit sum is to check if a number may be divided completely by numbers such as 9, 12, or 18.

I. The Digit Sum Test for Divisibility by 3 and 9.
If the digit sum or digit root of a number may be divided by 3 (or 9) then the number itself may be divided by 3 (or 9).
For example, 12, 111, 101010 and 300100200111 all have digit sums that may be divided by 3, therefore all of them may be divided by 3 evenly. However only 3001002000111, whose digit sum is 9, is divisible by 9.
Example A. Identify which of the following numbers are divisible by 3 and which are divisible by 9 by inspection.
a. 2345  b. 356004  c. 6312  d. 870490
Some Facts About Divisibility
We refer the above digit–sum check for 3 and 9 as **test I**.
We continue with **test II and III**.

**II. The Test for Divisibility by 2, 4, and 8**
A number is divisible by 2 if its last digit is even.
A number is divisible by 4 if its *last 2 digits* is divisible by 4 – you may ignore all the digits in front of them.
Hence ****32 is divisible by 4 but ****42 is not.
A number is divisible by 8 if its *last 3 digits* is divisible by 8.
Hence ****880 is divisible by 8, but ****820 is not.

**III. The Test for Divisibility by 5**
A number is divisible by 5 if its last digit is 5 or 0.
From the above checks, we get the following checks for important numbers such as 6, 12, 15, 18, 36, etc.
The idea is to do multiple checks on any given numbers.

---

Some Facts About Divisibility

**The Multiple Checks Principle**
If a number passes two different of tests I, II, or III, then it's divisible by the product of the numbers tested.
The number 102 is divisible by 3-via the digit sum test.
It is divisible by 2 because the last digit is even.
Hence 102 is divisible by 2\( \times 3 = 6 \times 17 \).
Example B. A bag contains 384 pieces of chocolate, can they be evenly divided by 12 kids? How about 2,349,876,543,214 pieces of chocolate with 18 kids?
For 384 its digit sum is 15 so it's divisible by 3 (but not 9).
Its last two digits are 84 which is divisible by 4. Hence 384 is divisible by 3 \( \times 4 \) or 12.
For 2,349,876,543,210 for 18, since it's divisible by 2, we only have to test divisibility for 9. Instead of actually find the digit sum, let's cross out the digits sum to multiple of 9. We see that it's divisible by 9, hence it's divisible by 18.
Some Facts About Divisibility

Ex. Check each for divisibility by 3 or 6 by inspection.
1. 106  2. 204  3. 402  4. 1134  5. 11340

Check each for divisibility by 4 or 8 by inspection.

11. Which numbers in problems 1 – 10 are divisible by 9?
12. Which numbers in problems 1 – 10 are divisible by 12?
13. Which numbers in problems 1 – 10 are divisible by 18?
1–s1 Signed Numbers

We track the “directions” of measurements by giving them positive (+) or negative (-) sign. Signed measurements represent amounts of increases versus decreases, surpluses versus deficiencies, credits versus debits and so on. Numbers with signs are called signed numbers.

Example A.

a. We deposited $400 into a bank account then withdrew $350 from the account, write the transactions using signed numbers. How much is left in the account?

Using “+” for deposit or having surplus in the account and “−” for withdraw or debit from the account, the transactions may be listed as: +400, −350. (In this section we will use red color for negative numbers for emphasis.)

The balance of the account is a surplus of $50 or +50.

We write the entire transactions as +400 − 350 = +50.

b. We deposited $400 into the account then withdrew $600 from the account, write the transactions using signed numbers. How much is left in the account?

The transactions may be listed as: +400, −600.

The account is short by $200. This is a deficiency so its −200.

We write the entire transactions as +400 − 600 = −200.

c. We deposited $400 in a bank account, later deposited another $200, then withdrew $350, then deposited $250, then withdrew $600, make a list of the transactions using signed numbers. How much is left?

The transactions are listed as +400, +200, −350, +250, −600. To find the final balance in the account, we note that the total of the deposits is +850 and the total of the withdrawals is −950, so the account is short of $100, or there is “−100” left in the account. We write these transactions as

+400 + 200 − 350 + 250 − 600 = −100.
Signed Numbers

The above operation of totaling two or more signed numbers into a single signed number is called the combining operation.

Example B.
\[+100 + 200 = +300, \quad -100 - 200 = -300\]
\[+500 - 300 = +200, \quad -500 + 300 = -200\]

A number written without a sign is treated as a positive number. Therefore, 100 + 200 is the same as +100 + 200 and both combined to 300.

In order to state precisely the rules for combining signed numbers, we introduce the notion of absolute values. In example A of the bank account, if we are only interested in the amount of the transactions but not the type of transactions, this amount is called the absolute value.

The absolute value of a number x is written as |x|.
Hence, |500| = 500, |−350| = 350, |−600| = 600, etc..

---

Signed Numbers

Rules for Combining Signed Numbers

I. To combine two or more numbers of the same signs, keep the sign, sum of the absolute values of the numbers.
\[+100 + 200 = +300, \quad -100 - 200 = -300\]

II. To combine two numbers of different signs, keep the sign of the number with larger absolute value, take the difference of the absolute values of the numbers.
\[+500 - 300 = +200, \quad -500 + 300 = -200\]

There are different ways to combine multiple signed numbers. We may combine them from left to right.

Example C. \[8 - 9 + 11 = -1 + 11 = 10\]

To combine many numbers, an alternative way is to do it as in example A of the bank transactions. That is, we combined all the positive ones (deposits) first, then combine all the negative ones (withdrawals), then combine the two results.
Signed Numbers

* The above method is easier when summing many numbers. * When doing this, it helps to move all the positive ones to the front and the negative ones to the back.

Example D.
\[ 7 - 11 + 14 - 12 + 15 - 19 - 6 - 11 \]
positive ones to the front
\[ = 7 + 14 + 15 - 11 - 12 - 19 - 6 - 11 \]
\[ = \frac{36}{\text{61}} = -25 \]

Another method for combining many signed numbers is to do two in groups of two’s. Hence
\[ 7 - 11 + 14 - 12 + 15 - 19 - 8 - 11 \]
group them in pairs
\[ = -4 + 2 -4 -19 \]
in pairs again
\[ = -2 -23 = -25 \]

(Two Method Strategy) Do it two ways to double check the answer when combining multiple signed numbers.

Signed Numbers

Exercise A. Combine
1. 2 + 3  2. 10 + 6  3. 34 + 21 + 4 + 17  4. –6 –2
8. –3 + 2  9. 5 –11  10. –14 + 15
11. 26 –15  12. 12 –13  13. –23 +18

B. Combine by moving the positive numbers to the front first. Combine the positive numbers, the negative numbers separately then then combine the two results.

14. 23 – 18 +7 –12  15. –6 –2 + 10 + 6
20. –4 + 7 – 23 + 8 + 17 – 8 + 6 + 9 – 22 – 2
21. Try to get the same answer for #20 by combining two numbers at a time without separating the positive numbers from the negative numbers.
1–s2 Addition/Subtraction of Signed Numbers

**Addition of Signed Numbers**

Adding signed numbers is the same as combining the numbers.

**Rule for Addition of Signed Numbers:**
To add two signed numbers, remove the parenthesis and combine the numbers, that is,

\[ a + (+b) = a + b, \quad a + (-b) = a - b \]

**Example A. Remove parentheses then combine.**

a. \[ 5 + (+4) \leftarrow \text{remove } "(\)" \]
   \[ = 5 + 4 = 9 \]

b. \[ -7 + (3) \leftarrow \text{remove } "(\)" \]
   \[ = -7 + 3 = -4 \]

c. \[ 2 + (-6) \leftarrow \text{remove } "(\)" \]
   \[ = 2 - 6 = -4 \]

d. \[ -4 + (-8) \leftarrow \text{remove } "(\)" \]
   \[ = -4 - 8 = -12 \]

---

**Addition and Subtraction of Signed Numbers**

e. \[ 5 + (4) + (-3) + (-8) + 4 - 6 \leftarrow \text{remove } "(\)"s \]
   \[ = 5 + 4 - 3 - 8 + 4 - 6 \]
   \[ = 5 + 4 + 4 - 3 - 8 - 6 \]
   \[ = 13 - 17 = -4 \]

**Subtraction of Signed Numbers**

For subtraction of signed numbers, we need the notion of "opposite" numbers. The numbers \( x \) and \( -x \) are said to be the opposite or the negative of each other.

The opposite of \( x \) is \( -x \). The opposite of \( -x \) is \( -(-x) = x \).

So the opposite of 6 is \( -6 \), the opposite of \( -12 \) is \( -(-12) = 12 \).

Note that the opposite of a negative number is positive.

**Rule for Subtraction of Signed Numbers:**

To subtract a signed number \( x \), remove the parenthesis and combine with the opposite of \( x \), that is,

\[ a - (+b) = a - b \quad \text{and} \quad a - (-b) = a + b \]
**Addition and Subtraction of Signed Numbers**

Example B. Remove parentheses then combine.

a. \(5 - (+4) \leftrightarrow \text{remove "( )", change to opposite} \)
   \[= 5 - 4 = 1\]

b. \(3 - (-7) \leftrightarrow \text{remove "( )", change to opposite} \)
   \[= 3 + 7 = 10\]

c. \(-12 - (-5) \leftrightarrow \text{remove "( )", change to opposite} \)
   \[= -12 + 5 = -7\]

**Summary for removing parentheses for addition and subtraction of signed numbers:**

For addition: \(a + (+b) = a + b\) \hspace{1cm} \(a + (-b) = a - b\)

For subtraction: \(a - (+b) = a - b\) \hspace{1cm} \(a - (-b) = a + b\)

Example C. Remove the parentheses, then combine.

a. \(-6 - (-8) - (-2) - (9) \leftrightarrow \text{remove "( )"} \)
   \[= -6 + 8 + 2 - 9 \]
   \[= 8 + 2 - 6 - 9 = 10 - 15 = -5\]

---

**Addition and Subtraction of Signed Numbers**

b. \(2 + (-4) - (-8) - (5) + (-9) \leftrightarrow \text{remove the ( )'s} \)
   \[= 2 - 4 + 8 - 5 - 9 \]
   \[= 2 + 8 - 4 - 5 - 9 \]
   \[= 10 - 18 \]
   \[= -6\]

The ( ), [ ], or { } are grouping symbols. Each set of symbol must contain both the left-hand and right-hand parts and each encloses calculations that are to be done within the symbols.

Example D.

a. \(-6 - (8 - 9) \leftrightarrow \text{do the calculation inside the "( )"} \)
   \[= -6 - (-1) \leftrightarrow \text{remove parentheses} \]
   \[= -6 + 1 \]
   \[= -5\]

b. \((-6 - 8) - 9 \leftrightarrow \text{do the calculation inside the "( )"} \)
   \[= (-14) - 9 \leftrightarrow \text{remove parentheses} \]
   \[= -14 - 9 = -23\]
Addition and Subtraction of Signed Numbers

If there is a pair of grouping symbols inside another a pair of grouping symbols, the inner set is to be calculated first.

Example E.

\[ 2 - [-6 - (8 + 9)] \quad \text{do the calculation inside the \text{“( )”}} \]
\[= 2 - [-6 - (17)] \quad \text{remove parentheses} \]
\[= 2 - [-6 - 17] \quad \text{do the calculation inside the \text{“[ ]”}} \]
\[= 2 - [-23] \]
\[= 2 + 23 \]
\[= 25 \]

Addition and Subtraction of Signed Numbers

Exercise A. Drop the parentheses and write the expressions in a simpler form, then combine.

1. \(7 + (+3)\)  
2. \(7 + (-3)\)  
3. \(-7 + (3)\)  
4. \(-7 + (-3)\)  
5. \(-7 + (3) + 4\)  
6. \(-7 + (-3) - 4\)  
7. \(-17 + (-23) + 24\)  
8. \(16 + (-31) - 22\)  
9. \(-8 + (-15) - 9\)  
10. \(-8 + (-15) - 9 + (-3)\)  

Exercise B. Drop the parentheses after the subtraction (don’t forget to switch the sign) and write the expressions in a simpler form, then combine.

11. \(7 - (+3)\)  
12. \(7 - (-3)\)  
13. \(-7 - (3)\)  
14. \(-7 - (-3)\)  
15. \(-7 - (3) + 4\)  
16. \(-7 - (-3) - 4\)  
17. \(-17 - (-23) + 24\)  
18. \(16 - (+31) - 22\)  
19. \(-8 - (-15) - 9\)  
20. \(-8 - (-15) - (-19) - (+42) - 3\)
Addition and Subtraction of Signed Numbers

C. Drop the parentheses after the arithmetic operations and write the expressions in a simpler form, then combine.

21. $7 - (+3) + (-3)$
22. $7 - (-3) + (3)$
23. $-7 - (-3) + (-5)$
24. $-8 + (-6) - (-3) - (5)$
25. $17 - (-6) + 4 - (-12) - 11$
26. $-17 - (-23) + 4 + (-16) - 11$
27. $5 - (-13) - 41 - (-32) - 18 - (-41)$
28. $-8 - (-15) - 9 + 35 - 7 + (-25) - 10$

D. Work the problems starting from the inside. Make sure that you copy the ( )’s, [ ]’s correctly.

29. $7 - [+3 + (-3)]$
30. $[7 - (-3)] + (3)$
31. $-7 - [(-3) + (-5)]$
32. $-8 + [(-6) - (-3)] - 5$
33. $-3 + (-6) - [(-3) - 5]$
34. $-[17 - (-23)] - [(-16) - 11]$
35. $5 - [(-13) - 41] - [(-32) - 18] - (-41)$
36. $-[8 - (-15)] - [9 + 35] - [7 + (-25) - 10]$
1–s3 Multiplication/Division of Signed Numbers

Rule for Multiplication of Signed Numbers
To multiply two signed numbers, we multiply their absolute values and use the following rules for the sign of the product.

\[ (+ \cdot + = - \cdot - = +) \]
\[ (+ \cdot - = - \cdot + = -) \]

Two numbers with the same sign multiplied yield a positive product.
Two numbers with opposite signs multiplied yield a negative product.

Example A.
a. \((5 \cdot (-4)) = -20\)
b. \((-5 \cdot (-4)) = 20\)

In algebra, multiplication operations are indicated in many ways. We use the following rules to identify multiplication operations.

Multiplication and Division of Signed Numbers
• If there is no operation indicated between two quantities, the operation between them is multiplication, so \(xy\) means \(x \cdot y\).
• If there is no operation indicated between a set of ( ) and a quantity, the operation between them is multiplication.
Hence \(x(a + b) = x \cdot (a + b)\) and \((a + b)x = (a + b) \cdot x\).
• If there is no operation indicated between two sets of ( )'s, the operation between them is multiplication.
Hence \((x + y)(a + b) = (x + y) \cdot (a + b)\)
However, if there is a “+” or “−” sign between the ( ) and a quantity, then the operation is to combine.
Hence \(3(+5) = (+5)3 = 15\), but \(3 + (5) = (3) + 5 = 8\), and \(-5(-5) = (-5)(-5) = 25\), but \((-5) - 5 = -5 - (5) = -10\).
To multiply many signed numbers together, we always determine the sign of the product first: the sign of the product is determined by the Even–Odd Rules, then multiply just the (absolute values of the) numbers.
Multiplication and Division of Signed Numbers

Even-Odd Rule for the Sign of a Product
- If there are even number of negative numbers in the multiplication, the product is positive.
- If there are odd number of negative numbers in the multiplication, the product is negative.

Example B.

a. \(-1(-2) 2 (-1) = -4\)
   - three negative numbers, so the product is negative

b. \((-2)^4 = (-2)(-2)(-2)(-2) = 16\)
   - four negative numbers, so the product is positive

Fact: A quantity raised to an even power is always positive
i.e. \(x^{\text{even}}\) is always positive (except 0).

---

Multiplication and Division of Signed Numbers

In algebra, \(a \div b\) is written as \(a/b\) or \(\frac{a}{b}\).

Rule for the Sign of a Quotient
Division of signed numbers follows the same sign-rules for multiplications.

\[
\begin{align*}
\frac{+}{+} &= + \\
\frac{-}{-} &= + \\
\frac{-}{+} &= - \\
\frac{+}{-} &= -
\end{align*}
\]

Two numbers with the same sign divided yield a positive quotient.
Two numbers with opposite signs divided yield a negative quotient.

Example C.

a. \(\frac{20}{4} = \frac{-20}{-4} = 5\)

b. \(-20 \div 4 = 20 \div (-4) = -5\)

c. \(\frac{(-6)^2}{-4} = \frac{36}{-4} = -9\)
**Multiplication and Division of Signed Numbers**

The Even–Odd Rule applies to more length * and / operations problems.

**Example D. Simplify.**

\[
\frac{(-4)(6)(-1)(-3)}{(-2)(-5)12}
\]

five negative signs so the product is negative

\[
= -\frac{4(6)(3)}{2(5)(12)}
\]
simplify just the numbers

\[
= -\frac{3}{5}
\]

Various form of the Even–Odd Rule extend to algebra and geometry. It’s the basis of many decisions and conclusions in mathematics problems.

The following is an example of the two types of graphs there are due to this Even–Odd Rule. (Don’t worry about how they are produced.)

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**Multiplication and Division of Signed Numbers**

The Even Power Graphs vs. Odd Power Graphs of \( y = x^n \)

- **Even Power Graphs**
  - \( y = x^2 \)
  - \( y = x^4 \)

- **Odd Power Graphs**
  - \( y = x^3 \)
  - \( y = x^5 \)
Multiplication and Division of Signed Numbers

Exercise A. Calculate the following expressions.

Make sure that you interpret the operations correctly.

1. 3 – 3
2. 3(–3)
3. (3) – 3
4. (–3) – 3
5. –3(–3)
6. (–3)(–3)
7. (–3) – (–3)
8. (–3) – (–3)

B. Multiply. Determine the sign first.

9. 2(–3)
10. (–2)(–3)
11. (–1)(–2)(–3)
12. 2(–2)(–3)
13. (–2)(–2)(–2)
14. (–2)(–2)(–2)
15. (–1)(–2)(–2)(–2)
16. 2(–1)(3)(–1)(–2)

C. Simplify. Determine the sign and cancel first.

17. \( \frac{12}{-3} \)
18. \( \frac{-12}{-3} \)
19. \( \frac{-24}{-8} \)
20. \( \frac{24}{-12} \)
21. \( \frac{(2)(-6)}{-8} \)
22. \( \frac{(-18)(-6)}{-9} \)
23. \( \frac{(-9)(6)}{(12)(-3)} \)
24. \( \frac{(15)(-4)}{(-8)(-10)} \)
25. \( \frac{(-12)(-9)}{(-27)(15)} \)
26. \( \frac{(-2)(-6)(-1)}{(2)(-3)(-2)} \)
27. \( \frac{3(-5)(-4)}{(-2)(-1)(-2)} \)
28. \( \frac{(-2)(3)(-4)(5)(-6)}{(-3)(4)(-5)(6)(-7)} \)
1-s4 Order of Operations

If we have two $5-bills and two $10-bills, we have the total of 2(5) + 2(10) = 30 dollars. To get the correct answer 30, we multiply the 2 and the 5 and multiply the 2 and the 10 first, then we add the products 10 and 20.

If I have three $10-bills and you have four $10-bills, we have 3 + 4 = 7 $10-bills, and we have a total of (3 + 4)10 = 70 $.

In this case, we group the 3 + 4 in the “( )” to indicate that we are to add them first, then multiply the sum to 10.

This motivates us to set the rules for the order of operations.

Order of Operations (excluding raising power)
Given an arithmetic expression, we perform the operations in the following order:

1st. Do the operations within grouping symbols, starting with the innermost grouping symbol.
2nd. Do multiplications and divisions (from left to right).
3rd. Do additions and subtractions (from left to right).

---

Order of Operations

Example A.

a. 4(−8) + 3(5)
   = −32 + 15
   = −17

b. 4 + 3(5 + 2)
   = 4 + 3(7)
   = 4 + 21
   = 25

c. 9 − 2[7 − 3(6 + 1)]
   = 9 − 2[7 − 3(7)]
   = 9 − 2[7 − 21]
   = 9 − 2[−14]
   = 9 + 28
   = 37

(Don’t perform “4 + 3” or “9 − 2” in the above problems!!)
Order of Operations

Exercise: Don't do the part that you shouldn't do!
1. \(6 + 3(3 + 1)\)  
2. \(10 - 4(2 - 4)\)
3. \(5 + 2[3 + 2(1 + 2)]\)  
4. \(5 - 2[3 + 2(5 - 9)]\)
Ans: a. 18  b. 18  c. 23  d. 15

Coefficients
Starting with 0, adding \(N\) copies of \(x\)'s to 0 is written as \(Nx\):
\[
0 + x + x + x + \ldots + x = Nx
\]
\[
\text{N copies added}
\]
So \(0 = 0x\)
\[
0 + x = 1x
\]
\[
0 + x + x = 2x
\]
\[
0 + x + x + x = 3x
\]
\[
0 + x + x + x + x = 4x
\]
The number \(N\) of added copies is called the coefficient.
So the coefficient of \(3x\) is 3.
Similarly \(ab + ab + ab + ab\) is \(4ab\), with coefficient 4, and that \(3(x + y)\) is \((x + y) + (x + y) + (x + y)\). Note that \(0x = 0\).

---

Order of Operations

Exponents
Starting with 1, multiplying \(N\) copies of \(x\)'s to 1 is written as \(x^N\):
\[
1 \cdot x \cdot x \cdot x \ldots x = x^N
\]
\[
\text{N copies of x's}
\]
So \(1 = x^0\) (\(x \neq 0\))
\[
1 \cdot x = x^1
\]
\[
1 \cdot x \cdot x = x^2
\]
\[
1 \cdot x \cdot x \cdot x \cdot x = x^4
\]
The number of multiplied copies \(N\) of \(x^N\) is called the exponent.
So the exponent of \(x^3\) is 3.
An exponent applies only to the quantity directly under it.
So \(ab^2 = a^1b^2\) and that \((ab)^3 = ab \cdot ab \cdot ab\). Note that \(x^0 = 1\).
Order of Operations

Example B. (Exponential Notation)

a. Expand \((-3)^2\) and simplify the answer.
   The base is \((-3)\).
   Hence \((-3)^2\) is \((-3)(-3) = 9\)

b. Expand \(-3^2\)
   The base of the 2\(^{nd}\) power is 3.
   Hence \(-3^2\) means \(-(3\cdot3) = -9\)

c. Expand \((3\cdot2)^2\) and simplify the answer.
   The base for the 2\(^{nd}\) power is \((3\cdot2)\).
   Hence \((3\cdot2)^2\) is \((3\cdot2)(3\cdot2) = (6)(6) = 36\)

d. Expand \(3\cdot2^2\) and simplify the answer.
   The base for the 2\(^{nd}\) power is 2.
   Hence \(3\cdot2^2\) means \(3\cdot2\cdot2 = 12\)

e. Expand \((-3y)^3\) and simplify the answer.
   \((-3y)^3\)
   \[ = (-3y)(-3y)(-3y) \quad \text{(the product of three negatives number is negative)} \]
   \[ = -(3)(3)(3)(y)(y)(y) \]
   \[ = -27y^3 \]

From part b above, we see that the power is to be carried out before multiplication. Below is the complete rules of order of operations.

Order of Operations (PEMDAS)

1\(^{st}\). (Parenthesis) Do the operations within grouping symbols, starting with the innermost one.
2\(^{nd}\). (Exponents) Do the exponentiation
3\(^{rd}\). (Multiplication and Division) Do multiplications and divisions in order from left to right.
4\(^{th}\). (Addition and Subtraction) Do additions and subtractions in order from left to right.
Order of Operations

Example C. Order of Operations

a. \(5^2 - 3^2\)
   \[= 25 - 9\]
   \[= 16\]
b. \(- (5 - 3)^2\)
   \[= -(2)^2\]
   \[= -4\]
c. \(-2 \cdot 3^2 + (2 \cdot 3)^2\)
   \[= -29 + (6)^2\]
   \[= -18 + 36\]
   \[= 18\]
d. \(-3^2 - 5(3 - 6)^2\)
   \[= -9 - 5(-3)^2\]
   \[= -9 - 5(9)\]
   \[= -9 - 45 = -54\]

Exercise A. Calculate the following expressions.

Make sure that you interpret the operations correctly.

1. \(3(-3)\)
2. \((3) - 3\)
3. \(3 - 3(3)\)
4. \(3(-3) + 3\)
5. \(+3(-3)(+3)\)
6. \(3 + (-3)(+3)\)

B. Make sure that you don’t do the ± too early.

7. \(1 + 2(3)\)
8. \(4 - 5(6)\)
9. \(7 - 8(-9)\)
10. \(1 + 2(3 - 4)\)
11. \(5 - 6(7 - 8)\)
12. \((4 - 3)2 + 1\)
13. \([1 - 2(3 - 4)] - 2\)
14. \(5 + [5 + 6(7 - 8)](+5)\)
15. \(1 + 2[1 - 2(3 + 4)]\)
16. \(5 - 6[5 - 6(7 - 8)]\)
17. \(1 - 2[1 - 2(3 - 4)]\)
18. \(5 + 6[5 + 6(7 - 8)]\)
19. \((1 + 2)(1 - 2(3 + 4)]\)
20. \((5 - 6)[5 - 6(7 - 8)]\)
21. \(1 - 2(-3)(-4)\)
22. \((-5)(-6) - (-7)(-8)\)

C. Make sure that you apply the powers to the correct bases.

23. \((-2)^2\) and \(-2^2\)
24. \((-2)^3\) and \(-2^3\)
25. \((-2)^4\) and \(-2^4\)
26. \((-2)^5\) and \(-2^5\)
27. \(2 \cdot 3^2\)
28. \((2 \cdot 3)^2\)
Order of Operations

D. Make sure that you apply the powers to the correct bases.

29. \((2)^2 - 3(2) + 1\)  
30. \(3(-2)^2 + 4(-2) - 1\)
31. \(-2(3)^2 + 3(3) - 5\)  
32. \(-3(-1)^2 + 4(-1) - 4\)
33. \(3(-2)^3 - 4(-2)^2 - 1\)  
34. \((2)^3 - 3(2)^2 + 4(2) - 1\)
35. \(2(-1)^3 - 3(-1)^2 + 4(-1) - 1\)  
36. \(-3(-2)^3 - 4(-2)^2 - 4(-2) - 3\)

E. Calculate.

37. \((6 + 3)^2\)  
38. \(6^2 + 3^2\)  
39. \((-4 + 2)^3\)  
40. \((-4)^3 + (2)^3\)
41. \(7^2 - 4^2\)  
42. \((7 + 4)(7 - 4)\)
43. \((-5)^2 - 3^2\)  
44. \((-5 + 3)(-5 - 3)\)
45. \(5^3 - 3^3\)  
46. \((5 - 3)(5^2 + 5*3 + 3^2)\)
47. \(4^3 + 2^3\)  
48. \((4 + 2)(4^2 - 4*2 + 2^2)\)
49. \((3)^2 - 4(2)(3)\)  
50. \((3)^2 - 4(1)(-4)\)
51. \((-3)^2 - 4(-2)(3)\)  
52. \((-2)^2 - 4(-1)(-4)\)
53. \(\frac{7}{5} - (-5)\)  
54. \(\frac{8 - 2}{5 - 3}\)
55. \((-4) - (-8)\)  
56. \(\frac{(-7) - (-2)}{(-3) - (-6)}\)
1-s5 Variables and Evaluation

In mathematics we use symbols such as \( x, y \) and \( z \) to represent numbers. These symbols are called **variables** because their values change depending on the situation. We use variables and mathematics operations to make **expressions** which are calculation procedures.

For example, if an apple cost $2 and \( x \) represents the number of apples, then “\( 2x \)” is the expression for the cost for \( x \) apples. Suppose we have 6 apples, set \( x = 6 \) in the expression \( 2x \), we obtain \( 2(6) = 12 \) for the total cost.

The value “6” for \( x \) is called **input (value)**. The answer 12 is called the **output**. This process of replacing the variables with input value(s) and find the output is called **evaluation**.

Each variable can represent one specific measurement only. Suppose we need an expression for the total cost of apples and pears and \( x \) represents the number of apples, we must use a different letter, say \( y \), to represent the number of pears since they are two distinct measurements.

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**Variables and Evaluation**

When evaluating an expression, replace the variables with the input-values enclosed with \( (\quad) \)’s.

**Example A.**

a. Evaluate \( -x \) if \( x = -6 \).

   When evaluating, insert the input enclosed in a “( )”.

   Therefore, set \( x = (-6) \) we’ve
   \[ -x \rightarrow -(-6) = 6 \]

b. Evaluate \( -3x \) if \( x = -6 \).

   \[ -3x \rightarrow -3(-6) = 18 \]

c. Evaluate \( -2x^2 \) if \( x = 6 \).

   \[ -2x^2 \rightarrow -2(6)^2 = -2(36) = -72 \]

d. Evaluate \( -4xyz \) if \( x = -3, y = -2, z = -1 \).

   \[ -4(-3)(-2)(-1) = 24 \]
Variables and Evaluation

e. Evaluate \( x - y \) if \( x = -3, \ y = -5 \).
   \[ x - y \rightarrow (-3) - (-5) = -3 + 5 = 2 \]

f. Evaluate \( 3x^2 - y^2 \) if \( x = 2 \) and \( y = -3 \).
   Replace \( x \) by \( 2 \) and \( y \) by \(-3\) in the expression, we have
   \[ 3(2)^2 - (-3)^2 \]
   \[ = 3(4) - 9 \]
   \[ = 12 - 9 \]
   \[ = 3 \]

g. Evaluate \(-x^2 + (-8 - y)^2\) if \( x = 3 \) and \( y = -2 \).
   Replace \( x \) by \( 3 \), \( y \) by \(-2\) in the expression,
   \[ -(3)^2 + (-8 - (-2))^2 \]
   \[ = -9 + (-6)^2 \]
   \[ = -9 + 36 \]
   \[ = 27 \]

Exercise.
Evaluate,
A. \(-2x\) with the input
1. \( x = 3 \)
2. \( x = -3 \)
3. \( x = -5 \)
4. \( x = -1/2 \)
B. \(-y - 2x\) with the input
5. \( x = 3, \ y = 2 \)
6. \( x = -2, \ y = 3 \)
7. \( x = -1, \ y = -4 \)
8. \( x = 3, \ y = -6 \)
C. \((-x)^2\) with the input
9. \( x = 3 \)
10. \( x = -3 \)
11. \( x = -5 \)
12. \( x = -1/2 \)
D. \(-x^2\) with the input
13. \( x = -2 \)
14. \( x = -3 \)
15. \( x = -9 \)
16. \( x = -1/3 \)
E. \(-2x^3\) with the input
17. \( x = 3 \)
18. \( x = -2 \)
19. \( x = -1 \)
20. \( x = -1/2 \)
F. \(3x^2 - 2x - 1\) with the input
21. \( x = -4 \)
22. \( x = -2 \)
23. \( x = -1 \)
24. \( x = 1/2 \)
Variables and Evaluation

G. \(-2y^2 + 3x^2\) with the input
25. \(x = 3, y = 2\)  
26. \(x = -2, y = -3\)
27. \(x = -1, y = -4\)  
28. \(x = -1, y = -1/2\)

H. \(x^3 - 2x^2 + 2x - 1\) with the input
29. \(x = 1\)  
30. \(x = -1\)  
31. \(x = 2\)  
32. \(x = \frac{1}{2}\)

I. \(-\frac{b}{2a}\) with the input
33. \(a = -1, b = -2\)  
34. \(a = 2, b = -4\)
35. \(a = -2, b = -8\)  
36. \(a = 2, b = -12\)

J. \(b^2 - 4ac\) with the input
37. \(a = -2, b = 3, c = -5\)  
38. \(a = 4, b = -2, c = -2\)
39. \(a = -1, b = -2, c = -3\)  
40. \(a = 5, b = -4, c = 4\)

K. \(\frac{a-b}{c-d}\) with the input
41. \(a = 1, b = -2, c = 2, d = -2\)
42. \(a = -4, b = -2, c = -1, d = -4\)
43. \(a = -2, b = 3, c = -5, d = 0\)
44. \(a = -1, b = -2, c = -2, d = 14\)

L. \(\frac{(a-b)(b-c)}{(c-d)(d-a)}\) with the input
45. \(a = 1, b = -2, c = 2, d = 2\)
46. \(a = -4, b = -2, c = -1, d = -4\)
47. \(a = -2, b = 3, c = -5, d = 0\)
48. \(a = -1, b = -2, c = -2, d = 14\)

M. \(b^2 - a^2 - c^2\) if
49. \(a = -2, b = 3, c = -5\)
50. \(a = 4, b = -2, c = -2\)

N. \(b^2 - 4ac\) if
51. \(a = -2, b = 3, c = -5\)
52. \(a = 4, b = -2, c = -2\)
2–1 Expressions

Example A.
a. We order pizzas from Pizza Grande. Each pizza is $3. How much would it cost for 4 pizzas? How much for x pizzas? For 4 pizzas, it would cost $3 \cdot 4 = $12. For x pizzas it would cost $3 \cdot x = $3x.

b. There is a $10 delivery charge. How much would it cost us in total if we want the x pizzas delivered? In total, it would be $3x + 10 in $.

Formulas such as “3x” or “3x + 10” are called expressions in mathematics. Mathematical expressions are calculation procedures and they are written with numbers, variables, and operation symbols. Expressions calculate outcomes.

The simplest type of expressions are of the form $ax + b$ where a and b are numbers. These are called linear expressions. The expressions “3x” or “3x + 10” are linear, the expressions “$x^2 + 1$” or “1/x” are not linear.

Expressions

Combining Expressions

Given an expression, each addend is called a term. There are two terms in $ax + b$.

- the $x$-term
- the number term or the constant term

Just as 2 apples + 3 apples = 5 apples we may combine two $x$-terms. Hence $2x + 3x = 5x$.

We combine positive $x$-terms and negative $x$-terms the same way we combine signed numbers. Hence $-3x - 5x = -8x$.

The number-terms may be combined such as $3 - 8 = -5$.

However just as 2 apple + 3 banana = 2 apple + 3 banana, (i.e. they can’t be combined), $x$-terms can’t be combined with number terms because they are different type of items. Hence $2 + 3x$ stays as $2 + 3x$ (not 5x).
Expressions

Example B. Combine.
\[2x - 4 + 9 - 5x = 2x - 5x - 4 + 9 = -3x + 5\]

When multiplying another number with an x-term, we multiply the number to the coefficient. Hence, 
\[3(5x) = 5x + 5x + 5x = (3\cdot 5)x = 15x\]
and 
\[-2(-4x) = (-2)(-4)x = 8x\]

We may multiply a number with an expression and expand the result by the distributive law.

Distributive Law
\[A(B + C) = AB + AC = (B + C)A\]

Example C. Expand then simplify.
\[a. \quad -5(2x - 4) \quad \text{distribute the -5}\]
\[= -5(2x) - (-5)(4)\]
\[= -10x + 20\]

Expressions

b. \[3(2x - 4) + 2(4 - 5x) \quad \text{expand}\]
\[= 6x - 12 + 8 - 10x\]
\[= -4x - 4\]

Distributive law gives us the option of expanding the content of the parentheses i.e. extract items out of containers.

Example D. A store sells two types of gift boxes Regular and Deluxe. The Regular has 12 apples and 8 bananas, the Deluxe has 24 apples and 24 bananas. We have 3 boxes of Regular and 4 boxes of Deluxe. How many apples and bananas are there?

Let A stands for apple and B stands for banana, then Regular = (12A + 8B) and Deluxe = (24A + 24B).

Three boxes of Regular and four boxes of Deluxe is
\[3(12A + 8B) + 4(24A + 24B)\]
\[= 36A + 24B + 96A + 96B = 132A + 120B\]
Hence we have 132 apples and 120 bananas.
Expressions

We usually start with the innermost set of parentheses to simplify an expression that has multiple layers of parentheses. In mathematics, ( )'s, [ ]'s, and { }'s are often used to distinguish the layers. This can't be the case for calculators or software where [ ] and { } may have other meanings. Always simplify the content of a set of parentheses first before expanding it.

Example E. Expand.
\[-3(-3x - [5 - 2(-4x - 6)] - 4)\]
expand,
\[= -3(-3x - [5 + 8x + 12] - 4)\]
simplify,
\[= -3(-3x - [17 + 8x] - 4)\]
expand,
\[= -3(-3x - 17 - 8x - 4)\]
simplify,
\[= -3(-11x - 21)\]
expand,
\[= 33x + 63\]

Expressions

X-terms with fractional coefficients may be written in two ways, \(\frac{p}{q} x\) or \(\frac{px}{q}\) (but not as \(\frac{p}{qx}\)) since \(\frac{p}{q} x = \frac{p}{q} \cdot \frac{x}{1} = \frac{px}{q}\).

Hence \(\frac{2}{3} x\) is the same as \(\frac{2x}{3}\).

Example E. Evaluate \(\frac{4}{3} x\) if \(x = 6\).

Set \(x = 6\) in \(\frac{4}{3} x\), we get \(\frac{4}{3} (6)^2 = 8\).

We may use the multiplier method to combine fraction terms i.e. multiply the problem by the LCD and divide by the LCD.

Example F. Combine \(\frac{4}{3} x + \frac{5}{4} x\)

\[\frac{4}{3} x + \frac{5}{4} x\]
Multiply and divide by their LCD = 12,
\[= (\frac{4}{3} x + \frac{5}{4} x) \cdot \frac{12}{12}\]
expand and cancel the denominators,
\[= (4.4 + 5.3x) \cdot \frac{12}{12} = (16x + 15x) / 12 = \frac{31x}{12}\]
Expressions

X-terms with fractional coefficients may be written in two ways,

\( \frac{p}{q} x \) or \( \frac{px}{q} \) (but not as \( \frac{p}{qx} \)) since \( \frac{p}{q} x = \frac{p}{q} \cdot \frac{x}{1} = \frac{px}{q} \)

Hence \( \frac{2}{3} x \) is the same as \( \frac{2x}{3} \).

Example E. Evaluate \( \frac{4}{3} x \) if \( x = 6 \).

Set \( x = 6 \) in \( \frac{4}{3} x \), we get \( \frac{4}{3}(6) \) = 8

We may use the multiplier method to combine fraction terms i.e. multiply the problem by the LCD and divide by the LCD.

Example F. Combine \( \frac{4}{3} x + \frac{5}{4} x \)

\[ \frac{4}{3} x + \frac{5}{4} x \]

Multiply and divide by their LCD = 12,

\( = \left( \frac{4}{3} x + \frac{5}{4} x \right) \frac{12}{12} \) expand and cancel the denominators,

\( = \left( 4 \cdot 4 x + 5 \cdot 3x \right) / 12 = (16x + 15x) / 12 = \frac{31x}{12} \)
Expressions

Exercise A. Combine like terms and simplify the expressions.
1. $3x + 5x$  
2. $3x - 5x$  
3. $-3x - 5x$  
4. $-3x + 5x$
5. $3x + 5x + 4$  
6. $3x - 5 + 2x$  
7. $8 - 3x - 5$
8. $8 - 3x - 5 - x$  
9. $8x - 4x - 5 - 2x$  
10. $6 - 4x - 5x - 2$
11. $3A + 4B - 5A + 2B$  
12. $-8B + 4A - 9A - B$

B. Expand then simplify the expressions.
13. $3(x + 5)$  
14. $-3(x - 5)$  
15. $-4(-3x - 5)$  
16. $-3(6 + 5x)$
17. $3(x + 5) + 3(x - 5)$  
18. $3(x + 5) - 4(-3x - 5)$
19. $-9(x - 6) + 4(-3x + 5x)$  
20. $-12(4x + 5) - 4(-7 - 5x)$
21. $7(8 - 6x) + 4(-3x + 5)$  
22. $2(-14x + 5) - 4(-7x - 5)$
23. $-7(-8 - 6x) - 4(-3x - 5x)$  
24. $-2(-14x - 5) - 6(-9x - 2)$
25. $3(A + 4B) - 5(A + 2B)$  
26. $-8(B + 4A) + 9(2A - B)$
27. $11(A - 4B) - 2(A - 12B)$  
28. $-6(B - 7A) - 8(A - 4B)$

Expressions

C. Starting from the innermost ( ) expand and simplify.
29. $x + 2[6 + 4(-3 + 5x)]$  
30. $-5[x - 4(-7 - 5x)] + 6$
31. $8 - 2[4(-3x + 5) + 6x] + x$  
32. $-14x + 5[x - 4(-5x + 15)]$
33. $-7x + 3[8 - [6(x - 2) - 3] - 5x]$  
34. $-3[8 - 6(x - 2) - 3] - 5x - 5[x - 3(-5x + 4)]$
35. $4[5(3 - 2x) - 6x] - 3[x - 2]x - 3(-5x + 4)]$

D. Combine using the LCD-multiplication method
36. $\frac{2}{3} x + \frac{3}{4} x$  
37. $\frac{4}{3} x - \frac{3}{4} x$
38. $-\frac{5}{8} x + \frac{1}{6} x$  
39. $-\frac{3}{8} x - \frac{5}{6} x$

Expressions

41. As in example D with gift boxes Regular and Deluxe, the Regular contains 12 apples and 8 bananas, the Deluxe has 24 apples and 24 bananas. Joe has 6 Regular boxes and 8 Deluxe boxes. How many of each type of fruit does he have?

42. As in example D with gift boxes Regular and Deluxe, the Regular contains 12 apples and 8 bananas, the Deluxe has 24 apples and 24 bananas. For large orders we may ship them in crates or freight-containers where a crate contains 100 boxes Regular and 80 boxes Deluxe and a container holds 150 Regular boxes and 100 Deluxe boxes.
King Kong ordered 4 crates and 5 containers, how many of each type of fruit does King Kong have?
2–2 Linear Equations I

Expressions produce outputs. Equations recover inputs.

Example A.

a. We order pizzas from Pizza Grande. Each pizza is $3. There is $10 delivery charge. How much would it cost if we want x pizzas delivered?
For x pizzas it would cost $3x + $10 ($) in total.

To have them delivered, it would cost $10 in total.

b. Suppose the total is $34, how many pizzas did we order?
We backtrack the calculation by subtracting the $10 for delivery so the cost for the pizzas is $24, each pizza is $3 so we must have ordered 8 pizzas.

In symbols, we've the equation \(3x + 10 = 34\),

backtrack-calculation: \[
\begin{align*}
3x + 10 &= 34 \\
-10 &= -10 \\
3x &= 24 \\
\frac{3x}{3} &= \frac{24}{3} \\
x &= 8 \text{ (pizzas)}
\end{align*}
\]

Linear Equations I

In the above examples, the symbolic method to find solutions may seem unnecessarily cumbersome but for complicated problems, the symbolic versions are indispensable.

An equation is two expressions set equal to each other. Equations look like:

\[
\text{left expression} = \text{right expression}
\]

or

\[
\text{LHS} = \text{RHS}
\]

We want to solve equations, i.e. we want to find the value (or values) for the variable \(x\) such that it makes both sides equal. Such a value is called a solution of the equation.

In the example above \(3x + 10 = 34\) is an equations and \(x = 8\) is the solution for this equations because \(3(8) + 10\) is 34.

Where as we use an expression to calculate future outcomes, we use an equation to help us to backtrack from known outcomes to the original input \(x\), the solution for the equation.
Linear Equations I

A *linear equation* is an equation where both the expressions on both sides are linear expressions such as

\[3x + 10 = 34,\]  
\[8 = 4x - 6.\]

A linear equation does not contain any higher powers of \(x\) such as \(x^2, x^3\):

\[x^2 - 3x = 2x - 3\] is not a linear equation because of the \(x^2\).

Linear equations are the easy to solve, i.e. it’s easy to manipulate a linear equation, to backtrack the calculations, to reveal what \(x\) is. The easiest linear equations to solve are the single-step equations such as the following ones,

\[x - 3 = 12,\]
\[12 = x + 3,\]
\[3 \cdot x = 12,\]
\[12 = \frac{x}{3}.

All four equation are one-step equations.

---

Linear Equations I

**Basic principle for solving one-step-equations:**
To solve one-step-equations, isolate the \(x\) on one side by applying the opposite operation to both sides of the equation.

Example B. Solve for \(x\)

a. \(x - 3 = 12\)
   - Add 3 to both sides
   \[x = 15\]
   - check: \(15 - 3 \neq 12\)
   \[12 = 12\text{ (yes)}\]

b. \(x + 3 = -12\)
   - Subtract 3 from both sides
   \[x = -15\]
   - check: \(-15 + 3 \neq -12\)
   \[-12 = -12\text{ (yes)}\]

c. \(3x = 15\)
   - Both sides divided by 3
   \[x = 5\]
   - check: \(3(5) = 15\)
   \[15 = 15\text{ (yes)}\]
Linear Equations I

d. \(-\frac{x}{3} = -12\) Multiply both sides by 3

\((-\frac{x}{3} = -12) \times 3\)

\(x = -36\) Check: \(-\frac{36}{3} = -12\)

**Fact:** Given a linear equation if we \(\times, \div, +, -\), to both sides by the same quantity, the new equation will have the same solution.

Next we solve equations that require two steps. These are the ones that we have to collect the \(x\)-terms (or the number–terms) first with addition or subtraction, then multiply or divide to get \(x\).

**Example C. Solve for \(x\)**

a. \(4x - 6 = 30\) Collect the numbers by adding 6 to both sides

\[+6\]

\[4x = 36\]

\[\frac{4}{4} \div \frac{4}{4}\]

\[x = 9\]

(Check this is the right answer.)

---

**Linear Equations I**

**Example C. Solve for \(x\)**

b. \(x - 6 = 3x\) Collect the \(x\)'s by subtracting \(x\) from both sides

\[-x\]

\[=-6 = 2x\]

\[\frac{2}{2} \div \frac{2}{2}\]

\[-3 = x\]

In real-life, we encounter linear equations often.

**Example D.** To make a cheese sandwich, we use two slices of bread each containing 70 calories and slices of cheese where each slice of cheese is 90 calories.

a. How many calories are there in the sandwich with 2 slices of cheese?

There are 140 cal in the bread and \(2 \times 90 = 180\) cal to make a total of \(140 + 180 = 320\) calories in the cheese.

b. What is the expression that calculate the number of calories of a sandwich with \(x\) slices of cheese?

There are \(140 + 90x\) calories in the sandwich.
Linear Equations I

c. How many slices of cheese are there in a 500–cal sandwich?

The total calories $14 + 90x$ is 500, i.e.

$$140 + 90x = 500$$

Subtract 140 from both sides

$$90x = 360$$

Divide both sides by 90

$$x = 4$$

So there are 4 slices of cheese in a 500–cal sandwich.

The more general linear equations have the form

$$\#x \pm \# = \#x \pm \#,$$

where $\#$ can be any number. We solve it by following steps:

1. Add or subtract to move the $x$-term to one side of the equation and get: $\#x \pm \# = \#$ or $\# = \#x \pm \#$

2. Add or subtract the $\#$ to separate the number-term from the $x$-term to get: $\#x = \#$ or $\# = \#x$.

3. Divide or multiply to get $x$:

$$x = \text{solution} \quad \text{or} \quad \text{solution} = x$$

---

Linear Equations I

$$2x - 3(1 - 3x) = 3(x - 6) - 1$$

$$2x - 3 + 9x = 3x - 18 - 1$$

$$11x - 3 = 3x - 19$$

$$-3x = -16$$

$$x = -2$$

---

Linear Equations I

$$2x - 3(1 - 3x) = 3(x - 6) - 1$$

$$2x - 3 + 9x = 3x - 18 - 1$$

$$11x - 3 = 3x - 19$$

$$-3x = -16$$

$$x = -2$$
Linear Equations I

Exercise

A. Solve in one step by addition or subtraction.

1. $x + 2 = 3$
2. $x - 1 = -3$
3. $-3 = x - 5$
4. $x + 8 = -15$
5. $x - 2 = -1/2$
6. $\frac{2}{3} = x - \frac{1}{2}$

B. Solve in one step by multiplication or division.

7. $2x = 3$
8. $-3x = -1$
9. $-3 = -5x$
10. $8x = -15$
11. $-4 = \frac{x}{2}$
12. $7 = \frac{-x}{3}$
13. $\frac{-x}{3} = -4$
14. $7 = -x$
15. $-x = -7$

C. Solve by collecting the x's to one side first. (Remember to keep the x's positive.)

16. $x + 2 = 5 - 2x$
17. $2x - 1 = -x - 7$
18. $-x = x - 8$
19. $-x = 3 - 2x$
20. $-5x = 6 - 3x$
21. $-x + 2 = 3 + 2x$
22. $-3x - 1 = 3 - 6x$
23. $-x + 7 = 3 - 3x$
24. $-2x + 2 = 9 + x$

D. Solve for x by first simplifying the equations to the form of $\#x \pm \# = \#x \pm \#$.

25. $2(x + 2) = 5 - (x - 1)$
26. $3(x - 1) + 2 = -2x - 9$
27. $-2(x - 3) = 2(-x - 1) + 3x$
28. $-(x + 4) - 2 = 4(x - 1)$
29. $x + 2(x - 3) = 2(x - 1) - 2$
30. $-2(x - 3) + 3 = 2(x - 1) + 3x + 13$
31. $-(x + 4) - 2(x + 1) = 4(x - 1) - 2$
32. $x + 1 + 2(x - 3) = 2(x - 1) - (2 - 2x)$
33. $4 - 3(2 - 2x) = 2(4x + 1) - 14$
34. $5(x - 2) - 3(3 - x) = -3(x + 2) + 2(4x + 1)$
35. $-3(2 - 2x) + 3(3 - x) = 5(x - 1) + 2(2 - 3x)$
36. $6(2x - 5) - 4(3x + 2) = 2x + 6(-3x - 4) - 8$
2–3 Linear Equations II

In this section we make some remarks about solving equations. Recall that to solve linear equations we simplify each side of equation first. Then we separate the x's from the numbers and extract the value of a single x.

I. (Change-side-change-sign Rule)
When gathering the x's to one side of the equation and the number terms to the other side, just move terms across to other side and switch their signs (instead of adding or subtracting to remove terms).

\[
\begin{align*}
  x + a &= b & \quad \text{change-side} \\
  x &= b - a & \quad \text{change-sign} \\
  x - a &= b & \quad \text{or} \\
  x &= b + a & \quad \text{change-side} \\
  x &= b - a & \quad \text{change-sign}
\end{align*}
\]

There are two sides to group the x's, the right hand side or the left hand side. It's better to collect the x's to the side so that it ends up with positive x-term.

Example A. Solve.

a. \(-5 = x + 6\) 
   move +6 to the left, 
   \(-6 \quad -5 = x\) 
   it turns into \(-6\) 
   \(-11 = x\)

b. \(3(x + 2) - x = 2(-2 + x) + x\) 
   simplify each side 
   \(3x + 6 - x = -4 + 2x + x\) 
   move 2x and \(-4\) to the other sides, change their signs 
   \(2x + 6 = -4 + 3x\) 
   \(4 + 6 = 3x - 2x\) 
   \(10 = x\)

c. \(\frac{3}{4} + x = \frac{1}{2}\) 
   move \(\frac{3}{4}\) to the other side, 
   change its sign 
   \(x = \frac{1}{2} - \frac{3}{4}\) 
   \(x = \frac{2}{4} - \frac{3}{4}\) 
   \(x = \frac{-1}{4}\)
Linear Equations II

II. (The Opposite Rule) If \(-x = c\), then \(x = -c\)

Example B.

a. If \(-x = -15\)
   then \(x = 15\)

b. If \(-x = \frac{3}{4}\)
   then \(x = -\frac{3}{4}\)

Recall that an equation or a relation expressed using fractions may always be restated in a way without using fractions.

Example C. The value of \(\frac{2}{3}\) of an apple is the same as the value of \(\frac{1}{3}\) of an orange, restate this relation in whole numbers.

We've \(\frac{2}{3} A = \frac{3}{4} G\). LCD = 12, multiply 12 to both sides.

\[
(\frac{2}{3} A = \frac{3}{4} G) \cdot 12
\]

8A = 9G or the value of 8 apples equal to 9 oranges.

---

Linear Equations II

III. (Fractional Equations Rule) Multiply fractional equations by the LCD to remove the fractions first, then solve.

Example D. Solve the equations

a. \(\frac{3}{4} x = -6\)
   \((\frac{3}{4} x = -6) / 4\)
   \[3x = -24\]
   \[x = -8\]

   clear the denominator,
   multiply both sides by 4
   div by 3

b. \(\frac{1}{4} x + \frac{2}{3} = \frac{5}{6} x - \frac{3}{4}\)
   \((\frac{1}{4} x + \frac{2}{3} = \frac{5}{6} x - \frac{3}{4}) \cdot 12\)
   \[3x + 8 = 10x - 9\]
   \[9x + 8 = 10x - 9\]
   \[9x = 17\]
   \[x = \frac{17}{9}\]

   the LCD = 12, multiply it to both sides to clear the denominators
   move 3x to the right and -9 to the left and switch signs.
   div by 7

---


Linear Equations II

c. \( \frac{15}{100} (x - 20) = \frac{45}{100} x - 27 \)  multiply 100 to both sides to remove denominators

\[ \frac{15}{100} (x - 20) = \frac{45}{100} x - 27 \times \frac{100}{100} \]

\[ 15 (x - 20) = 45x - 2700 \]
\[ 15x - 300 = 45x - 2700 \]
\[ 2700 - 300 = 45x - 15x \]
\[ 2400 = 30x \]
\[ 2400 / 30 = x \]
\[ 80 = x \]

d. \( 0.25(x - 100) = 0.10x - 1 \)

Change the decimal into fractions, we get

\[ \frac{25}{100} (x - 100) = \frac{10}{100} x - 1 \] multiply 100 to both sides to remove denominators

You finish it...

Ans: \( x = 160 \)

---

Linear Equations II

IV. (Reduction Rule) Use division to reduce equations to simpler ones.
Specifically divide the common factor of the coefficients of each term to make them smaller and easier to work with.

Example E. Simplify the equation first then solve.

\[ 14x - 49 = 70x - 98 \]

divide each term by 7

\[ \frac{14x - 49}{7} = \frac{70x - 98}{7} \]

\[ 2x - 7 = 10x - 14 \]

\[ 14 - 7 = 10x - 2x \]

\[ \frac{7}{8} = 8x \]
\[ \frac{7}{8} = x \]

We should always reduce the equation first if it is possible.
Linear Equations II

There’re two types of equations that give unusual results. The first type is referred to as identities. A simple identity is the equation \( x = x \). This corresponds to the trick question “What number \( x \) is equal to itself?” The answer of course is that \( x \) can be any number or that the solutions of equation \( x = x \) are all numbers.

This is also the case for any equation where both sides are identical such as \( 2x + 1 = 2x + 1, 1 - 4x = 1 - 4x \) etc...

An equation with identical expressions on both sides or can be rearranged into identical sides has all numbers as its solutions. Such an equation is called an identity.

**Example F. Solve.**

\[
\begin{align*}
2(x - 1) + 3 &= x - (-x -1) \quad \text{expand} \\
2x - 2 + 3 &= x + x + 1 \quad \text{simplify} \\
2x + 1 &= 2x +1 \quad \text{two sides are identical}
\end{align*}
\]

So this equation is an identity and every number is a solution.

---

Linear Equations II

At the opposite end of the identities are the “impossible” equations where there is no solution at all. An example is the equation \( x = x + 1 \). This corresponds to the trick question “What number is still the same after we add 1 to it?”

Of course there no such number.

If we attempt to solve \( x = x + 1 \)

\[
\begin{align*}
x - x &= 1 \\
0 &= 1
\end{align*}
\]

which is an impossibility. These equations are called *inconsistent equations*.

**Example F. Solve the equation**

\[
\begin{align*}
2(x - 1) + 4 &= x - (-x -1) \quad \text{expand} \\
2x - 2 + 4 &= x + x + 1 \quad \text{simplify} \\
2x + 2 &= 2x +1 \\
2 &= 1
\end{align*}
\]

So this is an inconsistent equation and there is no solution.
Linear Equations II

Exercise.
A. Solve for $x$ using the switch-side-switch-sign rule.
   Remember to move the $x$'s first and get positive $x$'s.

1. $x + 2 = 5 - 2x$
   \[ x + 5 = 5 \]
   \[ x = 0 \]

2. $2x - 1 = -x - 7$
   \[ 2x + x = 5 \]
   \[ 3x = 5 \]
   \[ x = \frac{5}{3} \]

3. $-x = x - 8$
   \[ -2x = 8 \]
   \[ x = -4 \]

4. $-x = 3 - 2x$
   \[ x = 3 \]

5. $-5x = 6 - 3x$
   \[ -2x = 6 \]
   \[ x = -3 \]

6. $-x + 2 = 3 + 2x$
   \[ 3x = -1 \]
   \[ x = -\frac{1}{3} \]

7. $-3x - 1 = 3 - 6x$
   \[ 3x - 6x = 4 \]
   \[ -3x = 4 \]
   \[ x = -\frac{4}{3} \]

8. $-x + 7 = 3 - 3x$
   \[ 2x = 4 \]
   \[ x = 2 \]

9. $-2x + 2 = 9 + x$
   \[ 3x = 7 \]
   \[ x = \frac{7}{3} \]

B. Solve the following fractional equations by using the LCD to remove the denominators first.

10. $\frac{3}{5}x = -2$
   \[ x = -\frac{10}{3} \]

11. $\frac{4}{3}x = -5$
   \[ x = -\frac{15}{4} \]

12. $\frac{3x}{4} = -\frac{1}{2}$
   \[ x = -\frac{2}{3} \]

13. $\frac{9x}{3} = \frac{3}{2}$
   \[ x = \frac{1}{2} \]

14. $\frac{-2}{3}x = -\frac{1}{2}$
   \[ x = \frac{3}{4} \]

15. $\frac{7x}{6} = -\frac{3}{4}$

16. $\frac{1}{6}x + \frac{2}{3} = \frac{5}{3}x - \frac{3}{2}$
   \[ x = \frac{3}{2} \]

17. $-\frac{3}{4}x - \frac{1}{6} = -\frac{5}{6}x - 1$
   \[ x = 6 \]

18. $\frac{3}{4}x - \frac{2}{5} = \frac{7}{10}x + \frac{3}{4}$
   \[ x = \frac{5}{2} \]

19. $-\frac{5}{8}x + \frac{7}{12} = -\frac{5}{16}x + 1$
   \[ x = \frac{2}{3} \]

20. $\frac{30}{100}(x - 20) = \frac{20}{100}x - 3$
   \[ x = 30 \]

21. $\frac{25}{100}(x + 5) - 3 = \frac{20}{100}(x - 5)$
   \[ x = 50 \]

22. $\frac{15}{100}(x + 15) = \frac{35}{100}x + 1$
   \[ x = 10 \]

23. $\frac{25}{100}(50 - x) + 2 = \frac{20}{100}(x - 50)$
   \[ x = 75 \]

24. $-0.3x - 0.25 = 1 - 0.6x$
   \[ 0.3x = 0.25 \]
   \[ x = \frac{5}{6} \]

25. $0.15x - 0.1x = 0.25x - 2.4$
   \[ 0.05x = 2.4 \]
   \[ x = 48 \]

26. $0.37 - 0.17x = 0.19x - 0.1$
   \[ 0.19x = 0.56 \]
   \[ x = 3 \]

27. $1.7x - 0.11 = 0.22 - 0.4x$
   \[ 2.1x = 0.33 \]
   \[ x = \frac{11}{66} \]

C. Reduce the equations then solve.

28. $-3x - 12 = 30 - 6x$
   \[ x = 22 \]

29. $15x - 10x = 25x - 20$
   \[ x = 4 \]

30. $-4(x - 3) = 12(x + 2) - 8x$
   \[ x = 10 \]

31. $15x - 10(x + 2) = 25x - 20$
   \[ x = -10 \]
Linear Equations II

D. Identify which equations are identities and which are inconsistent.

32. \(-x + 1 = -x + 2\)  
33. \(x + 2 = 5 - (3 - x)\)

34. \(-x + 1 = 5x - 2(3x + 1)\)  
35. \(-2(x + 2) = 5x - (7x - 4)\)

36. \(4(x - 3) - 2 = 1 - (14 - 4x)\)  
37. \(4(x - 3) - 2 = 1 - (15 - 4x)\)
2-4 Linear Word-Problems

The following are some key words translated into mathematical operations.

+: add, sum, plus, total, combine, increased by, # more than ..

−: subtract, difference, minus, decreased by, # less than ..

* : multiply, product, times, “fractions or %” of the amount ..

/: divide, quotient, shared equally, ratio ..

Twice = Double = 2-(amount)  Square = (amount)²

Example A. Let x be an unknown number, write the mathematical expressions represent the following amounts.

a. 300 more than twice x

“2x + 300”

2x

b. 2/3 of x

“ 2 - x ”

3

\( \frac{2}{3} \cdot x \)

c. 40% of the sum of x and 100

“ 40\% (x+100) ”

\( \frac{40}{100} \cdot (x+100) \)

e. Divide the difference between x-square and y-square, by 100.

\( \frac{x^2 - y^2}{100} \)

The word “difference” has two versions: x−y or y−x. One needs to clarify which version it’s in question before proceeding.

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Linear Word-Problems

Following are the steps to solve word problems.

• Read the whole problem and identify the unknown numbers.

• Select x to represent the unknown number where the other unknown numbers, if any, are stated in relation to it.

• Express the other unknowns in terms of the chosen x.

• Convert the numerical conclusion stated in the problem, into an equation-using the above expressions in x’s, then solve the equation for x.

Example B. Mary and Joe share $10.

Mary gets $9 more than Joe. How much does each get?

Let x = $ Joe gets, so Mary gets (x + 9). They’ve $10 in total.

\( x + (x + 9) = 10 \)

\( 2x + 9 = 10 \)

\( 2x = 10 - 9 \)

\( 2x = 1 \rightarrow x = \frac{1}{2} \) $ (for Joe)

and \( 9 + \frac{1}{2} = 9\frac{1}{2} \) $ (for Mary)

There are two unknown numbers.

Mary’s share is given in term of Joe’s share.

So Joe’s share is x.
Linear Word-Problems

Making Tables
If a problem provides the same types of information for multiple entities, organize the information into a table.

Example D. Put the following information into a table.

<table>
<thead>
<tr>
<th></th>
<th>No. of cats</th>
<th>No. of dogs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abe</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>Bob</td>
<td>15</td>
<td>25</td>
</tr>
</tbody>
</table>

b. Maria’s grocery list:
6 apples, 4 bananas, 3 cakes.

<table>
<thead>
<tr>
<th></th>
<th>Apple</th>
<th>Banana</th>
<th>Cake</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maria</td>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Don</td>
<td>2</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

By observing the above, one can easily appreciate the tables which make the numerical information easier to grasp.
Following examples utilize math–formulas of the form $A \times B = C$, and use tables in the construction of their solutions.

Linear Word-Problems

Cost Formula
The cost of purchasing multiple units of an item is given here, if

\[
p = \text{price per unit} \\
q = \text{quantity–in number of units} \\
C = \text{Cost of the purchase}
\]

then $p \times q = C$

Example E. a. Peanuts cost $5/lb and cashews cost $7/lb. We’ve 6 lbs of cashews and 8 lbs of peanuts. Put the data into a table which includes the cost of each item and the total cost.

<table>
<thead>
<tr>
<th></th>
<th>price/unit $</th>
<th>quantity of units</th>
<th>p\times q = Cost $</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peanuts</td>
<td>5</td>
<td>8</td>
<td>5*8 = 40</td>
</tr>
<tr>
<td>Cashews</td>
<td>7</td>
<td>6</td>
<td>7*6 = 42</td>
</tr>
</tbody>
</table>

**TOTAL: 82$**

b. Make the table if we have $x$ lbs of cashews and 2 more lbs of peanuts.

<table>
<thead>
<tr>
<th></th>
<th>price/unit $</th>
<th>quantity of units</th>
<th>p\times q = Cost $</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peanuts</td>
<td>5</td>
<td>(x + 2)</td>
<td>5(x + 2)</td>
</tr>
<tr>
<td>Cashews</td>
<td>7</td>
<td>x</td>
<td>7x</td>
</tr>
</tbody>
</table>

**TOTAL: 5(x + 2) + 7x → 12x + 10**
Linear Word-Problems

Now let’s turn the above into a problem.

Example F. a. Peanuts cost $5/lb and cashews cost $7/lb. We bought 2 more of lbs of peanuts than cashews and paid $82 in total. How many lbs of each did we buy?

Let \( x \) = # of lbs of cashews, then \((x + 2)\) = # of lbs of peanuts. Organize the information into a table.

<table>
<thead>
<tr>
<th></th>
<th>price/unit $</th>
<th>quantity of units</th>
<th>( p \cdot q ) = Cost $</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peanuts</td>
<td>5</td>
<td>((x + 2))</td>
<td>(5(x+2))</td>
</tr>
<tr>
<td>Cashews</td>
<td>7</td>
<td>(x)</td>
<td>(7x)</td>
</tr>
</tbody>
</table>

**TOTAL: 82$**

The total $82 is the sum of itemized costs:

\[
5(x + 2) + 7x = 82
\]

\[
12x + 10 = 82
\]

\[
12x = 72
\]

So there are 6 lb of cashews

\[
x = \frac{72}{12} = 6
\]

and 6 lb of peanuts.

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Linear Word-Problems

**Concentration Formula**
The concentration of a chemical in a solution is usually given in %. Let

\( r \) = concentration (in %)

\( S \) = amount of solution

\( C \) = amount of chemical

Then \( rS = C \)

Example G. We have 60 gallons brine (salt water) of 25% concentration. How much salt is there?

\[
r = \text{concentration} = 25\% = \frac{25}{100}
\]

\[
S = \text{amount of brine} = 60 \text{ gal}
\]

\[
C = \text{amount of salt} = ?
\]

Using \( rS = \frac{25}{100} \cdot 60 \)

\[
= \frac{25}{100} \cdot 60 = 15 \text{ gallons of salt.}
\]
Linear Word-Problems

Example H. (Mixture problem) How many gallons of 10% brine must be added to 30 gallons of 40% brine to dilute the mixture to 20% brine?

Make the standard table for mixture problems:

<table>
<thead>
<tr>
<th>concentration</th>
<th># of gallons</th>
<th>salt (rS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>x</td>
<td>$\frac{10}{100}x$</td>
</tr>
<tr>
<td>40%</td>
<td>30</td>
<td>$\frac{40}{100} \cdot 30$</td>
</tr>
<tr>
<td>20%</td>
<td>(x+30)</td>
<td>$\frac{20}{100} (x+30)$</td>
</tr>
</tbody>
</table>

Hence
\[
\frac{10}{100}x + \frac{40}{100} \cdot 30 = \frac{20}{100} (x + 30)
\]

Multiply by 100 to clear denominator:

\[
10x + 1200 = 20(x + 30) \\
10x + 1200 = 20x + 600 \\
-600 + 1200 = 20x - 10x \\
600 = 10x \Rightarrow 60 = x
\]

Ans: 60 gallons of 10% brine

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Linear Word-Problems

Simple Interest Formula

The interest rate of an investment is usually given in %. Let

- \( r \) = interest rate (in %)
- \( P \) = principal
- \( I \) = amount of interest for one year

Then \( rP = I \).

Example I. We have $3000 in a saving account that gives 6% annual interest. How much interest will be there after one year?

\[
r = \text{interest rate} = \frac{6}{100} \\
P = \text{principal} = 3000 \\
I = \text{amount of interest}
\]

Hence \( rP = \frac{6}{100} \cdot 3000 = 6(30) = $180 = \text{interest.} \)

Remark: The general formula for simple interest is \( P \cdot r \cdot t = I \) where \( t \) is the number of years. Note the similarity to the concentration formula.
Linear Word-Problems

Example J. (Mixed investments) We have $2000 more in a 6% account than in a 5% account. In one year, their combined interest is $560. How much is in each account?

We make a table. Let \( x \) = amount in the 5% account.

<table>
<thead>
<tr>
<th>rate</th>
<th>principal</th>
<th>interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>6%</td>
<td>( x + 2000 )</td>
<td>( \frac{6}{100}(x+2000) )</td>
</tr>
<tr>
<td>5%</td>
<td>( x )</td>
<td>( \frac{5}{100}x )</td>
</tr>
</tbody>
</table>

Total amount of interest is $560

Hence:

\[
\frac{6}{100}(x + 2000) + \frac{5}{100}x = 560
\]

\[
6(x + 2000) + 5x = 56000
\]

\[
6x + 12000 + 5x = 56000
\]

\[
11x = 56000 - 12000 = 44000
\]

\[
x = \frac{44000}{11} = 4000
\]

So we've $4000 in the 5% acct. and $6000 in the 6% acct.

Linear Word-Problems

Exercise A. Express each expression in terms of the given \( x \).

Let \( x = \$ \) that Joe has.

1. Mary has $10 less than what Joe has, how much does Mary have?
2. Mary has $10 more than twice what Joe has, how much does Mary have?
3. Mary has twice the amount that's $10 more than what Joe has, how much does Mary have?
4. Together Mary and Joe have $100, how much does Mary have?
5. Mary has $100 less than what Joe has, what is twice what Joe and Mary have together? Simplify your answer.
6. Mary has $100 more than Joe, what is half of what Joe and Mary have together? Simplify your answer.

B. Peanuts cost $3/lb, cashews cost $8/lb. We have \( x \) lbs of peanuts. Translate each quantity described below in terms of \( x \).

7. What is the cost for the peanuts?
8. We have 4 more lbs of cashews as peanuts, what is the cost for the cashews?
9. Refer to problem 7 and 8, what is the total cost for all the nuts? Simplify your answer.
10. We have twice as many of cashews as peanuts in weight, what is the total cost of the nuts? Simplify your answer.
11. We have a total of 50 lb of peanuts and cashews, how many lbs of cashews do we have? What is the total cost for all the nuts? Simplify you answers.
C. Money Sharing Problems. Select the x. Set up an equation then solve.
1. A and B are to share $120. A gets $30 more than B, how much does each get?
2. A and B are to share $120. A gets $30 more than twice what B gets, how much does each get?
3. A and B are to share $120. A gets $16 less than three times of what B gets, how much does each get?
4. A, B and C are to share $120. A gets $40 more than what B gets, C gets $10 less than what B gets, how much does each get?
5. A, B and C are to share $120. A gets $20 more than what twice of B gets. C gets $10 more than what A gets, how much does each get?
6. A and B are to share $120. A gets 2/3 of what B gets, how much does each get?
7. A, B and C are to share $120. A gets 1/3 of what B gets, C gets 1/3 of what B gets, how much does each get?

D. Wet Mixtures Problems (Make a table for each problem 8 – 10)
8. How many gallons of 20% brine must be added to 30 gallons of 40% brine to dilute the mixture to a 25% brine?
9. How many gallons of pure water must be added to 40 gallons of 30% brine to dilute the mixture to a 20% brine?
10. How many gallons of 10% brine must be added with 40% brine to make 30 gallons of 20% brine?
11. How many gallons of 50% brine must be added with pure water to make 30 gallons of 20% brine?

E. Mixed Investment Problems.
12. We have $3000 more saved in an account that gives 5% interest rate than in an account that gives 4% interest rate. In one year, their combined interest is $600. How much is in each account?
13. We have saved at a 5% account $2000 less than twice of in a 4% account. In one year, their combined interest is $1680. How much is in each account?
14. The combined saving in two accounts is $12,000. One account gives 5% interest rate, the other gives 4% interest rate. The combined interest is $570 in one year. How much is in each account?
15. The combined saving in two accounts is $15,000. One account gives 6% interest rate, the other gives 3% interest rate. The combined interest is $750 in one year. How much is in each account?

F. Dry Mixture Problems – These are similar to the wet-mixture problems.
16. Peanuts cost $3/lb, cashews cost $6/lb. How many lbs of peanuts are needed to mix with 12 lbs cashews to make a peanut–cashews–mixture that costs $4/lb?
17. Peanuts cost $3/lb, cashews cost $6/lb. How many lbs of each are needed to make 12 lbs of peanut–cashews–mixture that costs $4/lb?
18. Peanuts cost $2/lb, cashews cost $9/lb. How many lbs of cashews are needed to mix with 15 lbs of peanuts to get a peanut–cashews–mixture that costs $6/lb?
19. Peanuts cost $2/lb, cashews cost $8/lb. How many lbs of each are needed to make 50 lbs of peanut–cashews–mixture that cost $3/lb?
2–5 Literal Equations

Given an equation with many variables, the task is to solve for a particular variable. This means to isolate that variable to one side of the equation. We do this, just as solving equations in x, by +, –, *, / the same quantities to both sides of the equations. These quantities may be numbers or variables.

Example A.

a. Solve for x if \( x + b = c \)
   - Remove b from the LHS by subtracting from both sides
     \[ x + b = c \]
     \[ x = c - b \]

b. Solve for w if \( yw = 5 \)
   - Remove y from the LHS by dividing both sides by y.
     \[ yw = 5 \]
     \[ rac{yw}{y} = rac{5}{y} \]
     \[ w = \frac{5}{y} \]

Literal Equations

Adding or subtracting a term to both sides may be viewed as moving the term across the "=" and change its sign.
To solve for a specific variable in a simple literal equation, do the following steps.

1. If there are fractions in the equations, multiply by the LCD to clear the fractions.
2. Isolate the term containing the variable we wanted to solve for – move all the other terms to other side of the equation.
3. Isolate the specific variable by dividing the rest of the factor to the other side.

Example B.

a. Solve for x if \( (a + b)x = c \)
   - (a + b) \( x = c \) div the RHS by \( a + b \)
     \[ x = \frac{c}{a + b} \]
Literal Equations

b. Solve for w if \( 3y^2w = t - 3 \)
   \[
   3y^2w = t - 3 \quad \text{div the RHS by } 3y^2
   \]
   \[
   w = \frac{t - 3}{3y^2}
   \]

c. Solve for a if \( b^2 - 4ac = 5 \)
   \[
   b^2 - 4ac = 5 \quad \text{move } b^2 \text{ to the RHS}
   \]
   \[
   -4ac = 5 - b^2 \quad \text{div the RHS by } -4c
   \]
   \[
   a = \frac{5 - b^2}{-4c}
   \]

d. Solve for y if \( a(x - y) = 10 \)
   \[
   a(x - y) = 10 \quad \text{expand}
   \]
   \[
   ax - ay = 10 \quad \text{subtract ax}
   \]
   \[
   -ay = 10 - ax \quad \text{div by } -a
   \]
   \[
   y = \frac{10 - ax}{-a}
   \]

Literal Equations

Multiply by the LCD to get rid of the denominator then solve.

Example C.
Solve for d if \( -4 = \frac{3d + b}{d} \)

\[
-4 = \frac{3d + b}{d} \quad \text{multiply by the LCD } d
\]

\[
(-4 = 3d + b) \quad d
\]

\[
-4d = 3d + b \quad \text{move } -4d \text{ and } b
\]

\[
-b = 3d + 4d
\]

\[
-b = 7d \quad \text{div. by } 7
\]

\[
\frac{-b}{7} = d
\]
Literal Equations

Exercise. Solve for the indicated variables.
1. \( a - b = d - e \) for \( b \)  
2. \( a - b = d - e \) for \( e \)
3. \( 2b + d = e \) for \( b \)  
4. \( a^b + d = e \) for \( b \)
5. \( (2 + a)b + d = e \) for \( b \)  
6. \( 2L + 2W = P \) for \( W \)
7. \( (3x + 6)y = 5 \) for \( y \)  
8. \( 3x + 6y = 5 \) for \( y \)
9. \( 3x + 6xy = 5 \) for \( y \)  
10. \( 3x - (x + 6)y = 5z \) for \( y \)
11. \( w = \frac{t}{6} \) for \( t \)  
12. \( w = \frac{6}{t} \) for \( t \)
13. \( w = \frac{t - 3}{6} \) for \( t \)  
14. \( w = \frac{t - b}{a} \) for \( t \)
15. \( w = \frac{3t - b}{a} \) for \( t \)  
16. \( (3x + 6)y = 5 \) for \( x \)
17. \( w = \frac{t - 3}{6} + a \) for \( t \)  
18. \( w = \frac{at - b}{5} \) for \( t \)
19. \( w = \frac{at - b}{c} + d \) for \( t \)  
20. \( 3 = \frac{4t - b}{t} \) for \( t \)
2–7 Inequalities

We associate each real number with a position on a line, positive numbers to the right and negative numbers to the left.

This line with each position addressed by a real number is called the \textit{real (number) line}.

Given two numbers corresponding to two points on the real line, we define the number \textit{to the right to be greater than the number to the left}.

We write this as \( L < R \) and called this the \textit{natural form} because it corresponds to their respective positions on the real line. This relation may also be written as \( R > L \) (less preferable).
Inequalities

Example B.

a. Draw \(-1 \leq x < 3\).

It's in the natural form. Mark the numbers and \(x\) on the line in order accordingly.

\[ \begin{array}{c}
-1 \quad \bullet \quad 0 \quad \bullet \quad 3 \quad + \\
\end{array} \]

b. Draw \(0 > x > -3\).

Put it in the natural form \(-3 < x < 0\).

Then mark the numbers and \(x\) in order accordingly.

\[ \begin{array}{c}
-3 \quad \bullet \quad x \quad 0 \quad + \\
\end{array} \]

Expressions such as \(2 < x > 3\) or \(2 < x < -3\) do not have any solution.

Adding or subtracting the same quantity to both retains the inequality sign, i.e. if \(a < b\), then \(a \pm c < b \pm c\).

For example \(6 < 12\), then \(6 + 3 < 12 + 3\).

We use this fact to solve inequalities.

Inequalities

Example C. Solve \(x - 3 < 12\) and draw the solution.

\[ x - 3 < 12 \quad \text{add 3 to both sides} \]
\[ x - 3 + 3 < 12 + 3 \]
\[ x < 15 \]

A number \(c\) is positive means that \(0 < c\). We may multiply or divide a positive number to the inequality and keep the same inequality sign, i.e. if \(0 < c\) and \(a < b\), then \(ac < bc\).

For example \(6 < 12\) is true, then multiplying by \(3\)
\[ 3 \cdot 6 < 3 \cdot 12 \text{ or } 18 < 36 \text{ is also true.} \]

Example D. Solve \(3x \geq 12\) and draw the solution.

\[ 3x \geq 12 \quad \text{divide by 3 and keep the inequality sign} \]
\[ 3x/3 \geq 12/3 \]
\[ x \geq 4 \text{ or } 4 \leq x \]
Inequalities

A number $c$ is negative means $c < 0$. Multiplying or dividing by an negative number reverses the inequality sign, i.e. if $c < 0$ and $a < b$ then

$$ca > cb.$$  

For example $6 < 12$ is true. If we multiply $-1$ to both sides then

$$(-1)6 > (-1)12$$

$$-6 > -12$$ which is true.

Multiplying by $-1$ switches the left-right positions of the originals.

---

Example E. Solve $-x + 2 < 5$ and draw the solution.

$-x + 2 < 5$ subtract 2 from both sides

$-x < 3$ multiply by $-1$ to get $x$, reverse the inequality

$$-(-x) > -3$$

$x > -3$ or $-3 < x$

---

Inequalities

To solve inequalities:

1. Simplify both sides of the inequalities
2. Gather the $x$-terms to one side and the number-terms to the other sides (use the “change side-change sign” rule).
3. Multiply or divide to get $x$. If we multiply or divide by negative numbers to both sides, the inequality sign must be turned around. This rule can be avoided by keeping the $x$-term positive.

Example F. Solve $3x + 5 > x + 9$

$3x + 5 > x + 9$ move the $x$ and 5, change side-change sign

$3x - x > 9 - 5$

$2x > 4$ div. 2

$$
\frac{2x}{2} > \frac{4}{2}
$$

$x > 2$ or $2 < x$
Inequalities

Example G. Solve $3(2 - x) \geq 2(x + 9) - 2x$

$3(2 - x) \geq 2(x + 9) - 2x$ simplify each side
$6 - 3x \geq 2x + 18 - 2x$
$6 - 3x \geq 18$ move 18 and $-3x$ (change sign)
$6 - 18 \geq 3x$
$-12 \geq 3x$ div. by 3 (no need to switch $\geq$)
$\frac{-12}{3} \geq \frac{3x}{3}$
$-4 \geq x$ or $x \leq -4$

We also have inequalities in the form of intervals. We solve them by $+, -, \cdot, /$ to all three parts of the inequalities.
Again, we $+$ or $-$ remove the number term in the middle first, then divide or multiply to get $x$. The answer is an interval of numbers.

---

Inequalities

Example H. (Interval Inequality)

Solve $12 > -2x + 6 > -4$ and draw.

$12 > -2x + 6 > -4$ subtract 6
$-6$ $-6$ $-6$
$6 > -2x > -10$ div. by $-2$, switch inequality sign
$\frac{6}{-2} < \frac{-2x}{-2} < \frac{-10}{-2}$
$-3 < x < 5$

-3 0 5
Exercise. A. Draw the following Inequalities. Indicate clearly whether the end points are included or not.

1. \( x < 3 \)  
2. \( -5 \leq x \)  
3. \( x < -8 \)  
4. \( x \leq 12 \)

B. Write in the natural form then draw them.

5. \( x \geq 3 \)  
6. \( -5 > x \)  
7. \( x \geq -8 \)  
8. \( x > 12 \)

C. Draw the following intervals, state so if it is impossible.

9. \( 6 > x \geq 3 \)  
10. \( -5 < x \leq 2 \)  
11. \( 1 > x \geq -8 \)  
12. \( 4 < x \leq 2 \)

13. \( 6 > x \geq 8 \)  
14. \( -5 > x \leq 2 \)  
15. \( -7 \leq x \leq -3 \)  
16. \( -7 \leq x \leq -9 \)

D. Solve the following Inequalities and draw the solution.

17. \( x + 5 < 3 \)  
18. \( -5 \leq 2x + 3 \)  
19. \( 3x + 1 < -8 \)

20. \( 2x + 3 \leq 12 - x \)  
21. \( -3x + 5 \geq 1 - 4x \)

22. \( 2(x + 2) \geq 5 - (x - 1) \)  
23. \( 3(x - 1) + 2 \leq -2x - 9 \)

24. \( -2(x - 3) > 2(-x - 1) + 3x \)  
25. \( -(x + 4) - 2 \leq 4(x - 1) \)

26. \( x + 2(x - 3) < 2(x - 1) - 2 \)

27. \( -2(x - 3) + 3 \geq 2(x - 1) + 3x + 13 \)

E. Clear the denominator first then solve and draw the solution.

28. \( -4 \leq \frac{x}{2} \)  
29. \( 7 > \frac{-x}{3} \)  
30. \( \frac{-x}{5} < -4 \)

31. \( \frac{1}{2}x + \frac{2}{3} \geq \frac{2}{3}x \)  
32. \( \frac{-3}{4}x > \frac{-4}{3}x - 1 \)

33. \( \frac{3}{2}x - \frac{3}{8} \leq \frac{5}{4} \)  
34. \( -\frac{5}{8}x + \frac{7}{12} > 1 \)

35. \( -\frac{3}{2}x + \frac{2}{3} \leq \frac{4}{3}x - \frac{1}{4} \)  
36. \( \frac{5}{4}x + \frac{5}{6} < \frac{-1}{3}x - 2 \)

37. \( \frac{7}{12}x - \frac{3}{2} \geq \frac{1}{6}x - \frac{3}{4} \)

F. Solve the following interval inequalities.

38. \( -6 \leq 3x < 12 \)  
39. \( 6 > -2x > -4 \)

40. \( -2 < x + 2 < 5 \)  
41. \( -1 \geq 2x - 3 \geq -11 \)

42. \( -5 \leq 1 - 3x < 10 \)  
43. \( 11 > -1 - 3x > -7 \)
Comparison Statements, Inequalities and Intervals

“Positive” vs. “Negative”
A quantity $x$ is positive means that $0 < x$, and that $x$ is negative means that $x < 0$.

The phrase “the temperature $T$ is positive” is “$0 < T$”.

“Non–Positive” vs. “Non–Negative”
A quantity $x$ is non–positive means $x$ is not positive, or “$x \leq 0$”, and that $x$ is non–negative means $x$ is not negative, or “$0 \leq x$”.

The phrase “the account balance $A$ is non–negative” is “$0 \leq A$”.

“More/greater than” vs. “Less/smaller than”
Let $C$ be a number, $x$ is greater than $C$ means “$C < x$”, and that $x$ is less than $C$ means “$x < C$”. 

$\leftarrow x \text{ is less than } C$ 
$\leftarrow x \text{ is more than } C$
Comparison Statements, Inequalities and Intervals

“No more/greater than” vs “No less/smaller than”
and “At most” vs “At least”

A quantity $x$ is “no more/greater than $c$”
is the same as “$x$ is at most $c$” and means “$x \leq c$”.
A quantity $x$ is “no–less than $c$” is the same as
“$x$ is at least $c$” and means “$c \leq x$”.

The account balance $A$ is no–less than 500”
is the same as “$A$ is at least 500” or that “$500 \leq A$”.

The temperature $T$ is no–more than 250°
is the same as “$T$ is at most 250°” or that “$T \leq 250°$”.

Comparison Statements, Inequalities and Intervals

We also have the compound statements such as
“$x$ is more than $a$, but no more than $b$”.
In inequality notation, this is “$a < x \leq b$”.

and it’s denoted as: $(a, b]$
where “(, ]” means the end points are excluded
and that “[, ]” means the end points are included.
A line segment as such is called an interval.

Therefore the statement “the length $L$ of
the stick must be more than 5 feet but no
more than 7 feet” is “$5 < L \leq 7$”
or that $L$ must be in the interval $(5, 7]$.

Following is a list of interval notation.
Comparison Statements, Inequalities and Intervals

Let $a$, $b$ be two numbers such that $a < b$, we write

- $a \leq x \leq b$ as $[a, b]$.
- $a < x < b$ as $(a, b)$.
- $a \leq x < b$ as $[a, b)$.
- $a < x \leq b$ as $(a, b]$.

Using the $\infty$ symbol which means to “surpass all finite numbers”, we may write the rays

- $a \leq x$, as $[a, \infty)$.
- $a < x$, as $(a, \infty)$.
- $x \leq a$, as $(-\infty, a]$.
- $x < a$, as $(-\infty, a)$.

Comparison Statements, Inequalities and Intervals

Intersection and Union ($\cap$ & $\cup$) of Intervals

Let $I = [1, 3]$ as shown,

\[
\begin{array}{c}
I: \quad 1 & \quad 2 & \quad 3 \\
\end{array}
\]

and let $J = (2, 4)$ be another interval as shown,

\[
\begin{array}{c}
J: \quad 2 & \quad 3 & \quad 4 \\
\end{array}
\]

The common portion of the two intervals $I$ and $J$ shown here

\[
\begin{array}{c}
I: \quad 1 & \quad 2 & \quad 3 \\
J: \quad 2 & \quad 3 & \quad 4 \\
I \cap J: \quad 2 & \quad 3 \\
\end{array}
\]

is called the **intersection of $I$ and $J$**.

It's denoted as $I \cap J$ and this case $I \cap J = (2, 3)$. 

Comparison Statements, Inequalities and Intervals

The merge of the two intervals I and J shown here

I: \[\begin{array}{c}
1 \\
2 \\
3
\end{array}\]

J: \[\begin{array}{c}
2 \\
3 \\
4
\end{array}\]

I \cup J: \[\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}\]

is called the union of I and J and it’s denoted as I \cup J. In this case I \cup J = [1, 4).

Example A. Given intervals I, J, and K, perform the following set operation. Draw the solution and write the answer in the interval notation.

I: \[\begin{array}{c}
2 \\
1
\end{array}\] J: \(x > -2\) K: \(-3 < x \leq 1\)

a. K \cup J

We have \[\begin{array}{c}
-3 \\
-2 \\
1
\end{array}\] and K \cup J is the union: \[\begin{array}{c}
-3 \\
-2 \\
1
\end{array}\] so K \cup J = (-3, \infty).

---

Comparison Statements, Inequalities and Intervals

I: \[\begin{array}{c}
4 \\
3 \\
2 \\
1
\end{array}\] J: \(x > -2\) K: \(-3 < x \leq 1\)

b. K \cap I

We have \[\begin{array}{c}
-1 \\
0 \\
1
\end{array}\] The intersection is the overlapping portion as shown so K \cap I is \[\begin{array}{c}
-1 \\
0
\end{array}\] or (-3, -1).

Example B. Abe and Bob work at the same shop. Abe works after 2 pm till no more than 5 pm, Bob works from exactly 4 pm till before 7 pm, a. draw each person’s schedule on a time line and write them using the interval notation.

Abe schedule: \[\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8
\end{array}\] \(A: (2, 5]\)

Bob’s schedule: \[\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8
\end{array}\] \(B: [4, 7)\)
b. When will there someone working at the shop? 
Stack the schedules as shown.
The answer is the union of A and B 
i.e. \( A \cup B = (2, 7) \).
So there will be someone after 2 pm till before 7 pm.

b. When will both be working at the shop?
The time both persons be working is the intersection of their schedule, 
i.e. \( A \cap B = [4, 5] \). So both be there from 4 pm to 5 pm.

Comparison Statements, Inequalities and Intervals
The interval \([a, a]\) consists of one point \(\{x = a\}\). 
The empty set which contains nothing is denoted as \(\emptyset = \{\}\) and interval \((a, a) = (a, a) = [a, a) = \emptyset\).
Comparison Statements, Inequalities and Intervals

Exercise. A. Draw the following Inequalities. Translate each inequality into an English phrase. (There might be more than one way to do it)
1. $x < 3$  
2. $-5 \leq x$  
3. $x < -8$  
4. $x \leq 12$
5. $x \geq 3$  
6. $-5 > x$  
7. $x \geq -8$  
8. $x > 12$

Exercise. B. Translate each English phrase into an inequality. Draw the Inequalities.
Let $P$ be the number of people on a bus.
9. There were at least 50 people on the bus.
10. There were no more than 50 people on the bus.
11. There were less than 30 people on the bus.
12. There were no less than 28 people on the bus.
Let $T$ be temperature outside.
13. The temperature is no more than $-2^\circ$.
14. The temperature is at least $35^\circ$.
15. The temperature is positive.

Comparison Statements, Inequalities and Intervals

Let $M$ be the amount of money I have.
16. I have at most $25.$
17. I have a non–positive amount of money.
18. I have less than $45.$
19. I have at least $250.$

Let the basement floor number be given as a negative number and let $F$ be the floor number that we are on.
20. We are below the $7^{\text{th}}$ floor.
21. We are above the first floor.
22. We are not below the $3^{\text{rd}}$ floor basement.
23. We are on at least the $45^{\text{th}}$ floor.
24. We are between the $4^{\text{th}}$ floor basement and the $10^{\text{th}}$ floor.
25. We are in the basement.
Comparison Statements, Inequalities and Intervals

27. Apu, Bobo, and Chuck work at the same shop.
Apu works from 1 pm till no more than 6 pm,
Bobo works from 4 pm till before 9 pm,
Chuck works after 3 pm till no later than 10 pm.
Let A, B, and C be the intervals that represent the time that
Apu, Bobo, and Chuck work respectively, draw A, B, and C.

28. Using the A, B, and C from 27, translate the following sets
into English, draw the interval and write answers in the interval
notation.

a. A ∪ B  
  d. (A ∪ B) ∪ C  
  g. (A ∩ B) ∪ C
b. C ∩ B  
  e. (A ∩ C) ∩ B
  f. (A ∪ B) ∩ C
A coordinate system is a system of assigning addresses for positions in the plane (2D) or in space (3D). The rectangular coordinate system for the plane consists of a rectangular grid where each point in the plane is addressed by an ordered pair of numbers $(x, y)$.

The horizontal axis is called the $x$-axis. The vertical axis is called the $y$-axis. The point where the axes meet is called the origin. Starting from the origin, each point is addressed by its ordered pair $(x, y)$ where:

- $x =$ amount to move right (+) or left (−).
- $y =$ amount to move up (+) or down (−).

Rectangular Coordinate System

The ordered pair $(x, y)$ corresponds to a point is called the coordinate of the point, $x$ is the x-coordinate and $y$ is the y-coordinate.

Example A.
Label the points
$A(-1, 2), B(-3, -2), C(0, -5)$.

Example B: Find the coordinate of $P, Q, R$ as shown.
$P(4, 5), Q(3, -5), R(-6, 0)$
Rectangular Coordinate System

The coordinate of the origin is (0, 0).

Any point on the x-axis has coordinate of the form (x, 0).

Any point on the y-axis has coordinate of the form (0, y).

The axes divide the plane into four parts. Counter clockwise, they are denoted as quadrants I, II, III, and IV. Respectively, the signs of the coordinates of each quadrant are shown.

Rectangular Coordinate System

When the x-coordinate of the a point (x, y) is changed to its opposite as (−x, y), the new point is the reflection across the y-axis.

When the y-coordinate of the a point (x, y) is changed to its opposite as (x, −y), the new point is the reflection across the x-axis. (−x, −y) is the reflection of (x, y) across the origin.
Rectangular Coordinate System

**Movements and Coordinates**

Let A be the point (2, 3).
Suppose its \textit{x–coordinate is increased by 4} to
\[(2 + 4, 3) = (6, 3)\] - to the point B,
this corresponds to \textit{moving A to the right by 4}.

Similarly if the \textit{x–coordinate of (2, 3) is decreased by 4} to
\[(2 - 4, 3) = (-2, 3)\] - to the point C,
this corresponds to \textit{moving A to the left by 4}.

Hence we conclude that changes in the \textit{x–coordinates} of a point
move the point right and left.
If the \textit{x–change is +}, the point moves to the right.
If the \textit{x–change is –}, the point moves to the left.

Rectangular Coordinate System

Again let A be the point (2, 3).
Suppose its \textit{y–coordinate is increased by 4} to
\[(2, 3 + 4) = (2, 7)\] - to the point D,
this corresponds to \textit{moving A up by 4}.

Similarly if the \textit{y–coordinate of (2, 3) is decreased by 4} to
\[(2, 3 - 4) = (2, -1)\] - to the point E,
this corresponds to \textit{moving A down by 4}.

Hence we conclude that changes in the \textit{y–coordinates} of a point
move the point right and left.
If the \textit{y–change is +}, the point moves up.
If the \textit{y–change is –}, the point moves down.
Rectangular Coordinate System

Example. C.

a. Let A be the point \((-2, 4)\). What is the coordinate of the point B that is 100 units directly left of A?

Moving left corresponds to decreasing the x-coordinate.
Hence B is \((-2 - 100, 4) = (-102, 4)\).

b. What is the coordinate of the point C that is 100 units directly above A?

Moving up corresponds to increasing the y-coordinate.
Hence C is \((-2, 4 +100) = (-2, 104)\).

c. What is the coordinate of the point D that is 50 to the right and 30 below A?

Here is the vertical format for the calculation:

Adding 50 to the x-coordinate to move right, and -30 to the y-coordinate to move down.

Hence D has the coordinate \((-2 + 50, 4 - 30) = (48, -26)\).

Rectangular Coordinate System

d. The point A\((-2, 4)\) is 50 to the right and 30 below the point E. What's the coordinate of the point E?

Let the coordinate of E be \((a, b)\).
In the vertical format we have

\[
\begin{align*}
\text{Hence } a + 50 &= -2 \text{ so } a = -52 \\
\text{and that } b + (-30) &= 4 \text{ so } b = 34.
\end{align*}
\]

Hence E is \((-52, 34)\).
Rectangular Coordinate System

Example. C.

a. Let A be the point (−2, 4). What is the coordinate of the point B that is 100 units directly left of A?

Moving left corresponds to decreasing the x-coordinate. Hence B is (−2 − 100, 4) = (−102, 4).

b. What is the coordinate of the point C that is 100 units directly above A?

Moving up corresponds to increasing the y-coordinate. Hence C is (−2, 4) = (−2, 4 +100) = (−2, 104).

c. What is the coordinate of the point D that is 50 to the right and 30 below A?

We need to add 50 to the x-coordinate (to the right) and subtract 30 from the y-coordinate (to go down). Hence D has coordinate (−2 + 50, 4 − 30) = (48, −26).

Rectangular Coordinate System

Exercise. A.

a. Write down the coordinates of the following points.
Rectangular Coordinate System

Ex. B. Plot the following points on the graph paper.
2. a. (2, 0)    b. (−2, 0)    c. (5, 0)    d. (−8, 0)    e. (−10, 0)
All these points are on which axis?
3. a. (0, 2)    b. (0, −2)    c. (0, 5)    d. (0, −6)    e. (0, 7)
All these points are on which quadrant?
4. a. (5, 2)    b. (2, 5)    c. (1, 7)    d. (7, 1)    e. (6, 6)
All these points are in which quadrant?
5. a. (−5, −2)    b. (−2, −5)    c. (−1, −7)    d. (−7, −1)    e. (−6, −6)
All these points are in which quadrant?
6. List three coordinates whose locations are in the 2nd quadrant and plot them.
7. List three coordinates whose locations are in the 4th quadrant and plot them.

Rectangular Coordinate System

C. Find the coordinates of the following points. Draw both points for each problem.
The point that's
8. 5 units to the right of (3, −2).
9. 6 units to the right of (−4, 2).
10. 4 units to the left of (−1, −5).
11. 6 units to the left of (2, −6).
12. 3 units to the left and 6 units down from (−2, 5).
13. 1 unit to the right and 5 units up from (−3, 1).
14. 3 units to the right and 3 units down from (−3, 4).
15. 2 units to the left and 6 units up from (4, −1).
3–2 Linear Equations and Lines

We solved 1st degree (linear) equations such as $2x + 1 = 5$, which has a single variable $x$, to obtain its solution $x = 2$. We view this solution as the address of a position on a line and label it to produce a “picture” of the answer:

\[ \begin{array}{c}
5 \\
2 \\
\hline
x
\end{array} \]

The picture of $x = 2$

If we have a two–variable 1st degree equation such as

\[ 2x + y = 5 \]

then we are free to select $x$ and $y$. For instance $x = 2$ and $y = 1$ make the equation true.

By viewing $(2, 1)$ as the coordinate of a position in the xy-coordinate system, we have a picture of this solution. Having the liberty of choosing two numbers means there are many pairs of solutions, thus more solution–points can be plotted. These points form the graph of the equation.

---

Linear Equations and Lines

In the rectangular coordinate system, ordered pairs $(x, y)$'s correspond to locations of points. Collections of points may be specified by the mathematics relations between the $x$-coordinate and the $y$ coordinate. The plot of points that fit a given relation is called the graph of that relation. To make a graph of a given mathematics relation, make a table of points that fit the description and plot them.

Example A. Graph the points $(x, y)$ where $x = -4$ (y can be anything).

Make a table of ordered pairs of points that fit the description $x = -4$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>-4</td>
<td>2</td>
</tr>
<tr>
<td>-4</td>
<td>4</td>
</tr>
<tr>
<td>-4</td>
<td>6</td>
</tr>
</tbody>
</table>

Graph of $x = -4$
Example B. Graph the points \((x, y)\) where \(y = x\).
Make a table of points that fit the description \(y = x\). To find one such point, we set one of the coordinates to be a number, any number, than use the relation to find the other coordinate. Repeat this a few times.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Graph the points \((x, y)\) where \(y = x\)

---

**Linear Equations and Lines**

*First degree equation* in the variables \(x\) and \(y\) are equations that may be put into the form \(Ax + By = C\) where \(A, B, C\) are numbers. First degree equations are the same as *linear equations*. They are called *linear* because their graphs are *straight lines*. To graph a linear equation, find a few ordered pairs that fit the equation. To find one such ordered pair, assign a value to \(x\), plug it into the equation and solve for the \(y\) (or assign a value to \(y\) and solve for the \(x\)). For graphing lines, find at least two ordered pairs.

Example C.
Graph the following linear equations.

a. \(y = 2x - 5\)

Make a table by selecting a few numbers for \(x\). For easy calculation we set \(x = -1, 0, 1,\) and \(2\). Plug each of these values into \(x\) and find its corresponding \(y\) to form an ordered pair.
Linear Equations and Lines

For \( y = 2x - 5 \):
- If \( x = -1 \), then \( y = 2(-1) - 5 = -7 \)
- If \( x = 0 \), then \( y = 2(0) - 5 = -5 \)
- If \( x = 1 \), then \( y = 2(1) - 5 = -3 \)
- If \( x = 2 \), then \( y = 2(2) - 5 = -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-7</td>
</tr>
<tr>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Linear Equations and Lines

b. \(-3y = 12\)
Simplify as \( y = -4\)
Make a table by selecting a few numbers for \( x \).
However, \( y = -4 \) is always.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>6</td>
<td>-4</td>
</tr>
</tbody>
</table>

Linear Equations and Lines

c. \( 2x = 12 \)
Simplify as \( x = 6 \)
Make a table.
However the only selection for \( x \) is \( x = 6 \) and \( y \) could be any number.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
**Linear Equations and Lines**

Summary of the graphs of linear equations:

- **a.** \( y = 2x - 5 \)
  - If both variables \( x \) and \( y \) are present in the equation, the graph is a **tilted line**.

- **b.** \(-3y = 12\)
  - If the equation has only \( y \) (no \( x \)), the graph is a **horizontal line**.

- **c.** \( 2x = 12 \)
  - If the equation has only \( x \) (no \( y \)), the graph is a **vertical line**.

---

**Linear Equations and Lines**

The \( x \)-Intercepts is where the line crosses the \( x \)-axis. We set \( y = 0 \) in the equation to find the \( x \)-intercept.

The \( y \)-Intercepts is where the line crosses the \( y \)-axis. We set \( x = 0 \) in the equation to find the \( y \)-intercept.

Since two points determine a line, an easy method to graph linear equations is the *intercept method*, i.e. plot the \( x \)-intercept and the \( y \) intercept and the graph is the line that passes through them.

**Example C. Graph** \( 2x - 3y = 12 \) by the intercept method.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-4)</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

- If \( x = 0 \), we get \( 2(0) - 3y = 12 \) so \( y = -4 \)
- If \( y = 0 \), we get \( 2x - 3(0) = 12 \) so \( x = 6 \)
Linear Equations and Lines

Exercise. A. Solve the indicated variable for each equation with the given assigned value.
1. \(x + y = 3\) and \(x = -1\), find \(y\).
2. \(x - y = 3\) and \(y = -1\), find \(x\).
3. \(2x = 6\) and \(y = -1\), find \(x\).
4. \(-y = 3\) and \(x = 2\), find \(y\).
5. \(2y = 3 - x\) and \(x = -2\), find \(y\).
6. \(y = -x + 4\) and \(x = -4\), find \(y\).
7. \(2x - 3y = 1\) and \(y = 3\), find \(x\).
8. \(2x = 6 - 2y\) and \(y = -2\), find \(x\).
9. \(3y - 2 = 3x\) and \(x = 2\), find \(y\).
10. \(2x + 3y = 3\) and \(x = 0\), find \(y\).
11. \(2x + 3y = 3\) and \(y = 0\), find \(x\).
12. \(3x - 4y = 12\) and \(x = 0\), find \(y\).
13. \(3x - 4y = 12\) and \(y = 0\), find \(x\).
14. \(6 = 3x - 4y\) and \(y = -3\), find \(x\).

Linear Equations and Lines

B. a. Complete the tables for each equation with given values.
   b. Plot the points from the table. c. Graph the line.

<table>
<thead>
<tr>
<th>15. (x + y = 3)</th>
<th>16. (2y = 6)</th>
<th>17. (x = -6)</th>
<th>18. (y = x - 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td></td>
<td>-2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>19. (2x - y = 2)</th>
<th>20. (3y + 6 = 2x)</th>
<th>21. (y = -6)</th>
<th>22. (3y + 4x = 12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td></td>
<td>-2</td>
</tr>
</tbody>
</table>
Linear Equations and Lines

C. Make a table for each equation with at least 3 ordered pairs. (Remember that you get to select one entry in each row as shown in the tables above) then graph the line.

23. \( x - y = 3 \)  
24. \( 2x = 6 \)  
25. \( -y - 7 = 0 \)

26. \( 0 = 8 - 2x \)  
27. \( y = -x + 4 \)  
28. \( 2x - 3 = 6 \)

29. \( 2x = 6 - 2y \)  
30. \( 4y - 12 = 3x \)  
31. \( 2x + 3y = 3 \)

32. \( -6 = 3x - 2y \)  
33. \( 3x = 4y \)  
34. \( 5x + 2y = -10 \)

35. For problems 29, 30, 31 and 32, use the intercept tables as shown to graph the lines.

36. Why can't we use the above intercept method to graph the lines for problems 25, 26 or 33?

37. By inspection identify which equations give horizontal lines, which give vertical lines and which give tilted lines.
3–3 Systems of Linear Equations

If 2 hamburgers cost $4, then one hamburger is $2. We need one piece of information to solve for one unknown quantity. But if a hamburger and a drink cost $4 then we won't be able to pin down the price of each item—not enough information. In general, to find two unknowns, two pieces of information are needed, for three unknowns, three pieces of information, etc…

A system of linear equations is a collection two or more linear equations in two or more variables. A solution for the system is a collection of numbers, one for each variable, that works for all equations in the system.

For example,

\[
\begin{align*}
2x + y &= 7 \\
x + y &= 5
\end{align*}
\]

is a system and that \(x = 2\) and \(y = 3\) is a solution because the pair fits both equations. This solution is written as \((2, 3)\).

---

Systems of Linear Equations

Example A.
Suppose two hamburgers and a salad cost $7, and one hamburger and one salad cost $5.
Let \(x =\) cost of a hamburger, \(y =\) cost of a salad.
We can translate the information into the system

\[
\begin{align*}
2x + y &= 7 \\
x + y &= 5
\end{align*}
\]

The difference between the order is one hamburger and the difference between the cost is $2, so the hamburger is $2.
In symbols, subtract the equations, i.e. \(E1 - E2:\n
\[
\begin{align*}
2x + y &= 7 \\
-\) \(x + y &= 5
\end{align*}
\]

\(x + 0 = 2\) so \(x = 2\)
substituting 2 for \(x\) back into \(E2\), we get: \(2 + y = 5\)
or \(y = 3\)
Hence the solution is \((2, 3)\) or, $2 for a hamburger, $3 for a salad.
Example B.
Suppose four hamburgers and three salads is $18, three hamburger and two salads is $13. How much does each cost?
Let \( x \) = cost of a hamburger, \( y \) = cost of a salad.
We translate the information into the system:
\[
\begin{align*}
4x + 3y &= 18 \quad \text{------- E1} \\
3x + 2y &= 13 \quad \text{------- E2}
\end{align*}
\]
Subtracting the equations now will not eliminate the \( x \) nor \( y \).
We have to adjust the equations before we add or subtract them to eliminate \( x \) or \( y \).
Suppose we chose to eliminate the \( y \)-terms, we find the LCM of \( 3y \) and \( 2y \) first. It's \( 6y \). Multiplying E1 by \((-2)\), E2 by \(3\), and add the results, the \( y \)-terms are eliminated.
\[
\begin{align*}
\text{E1}\times(-2): \quad &-8x - 6y = -36 \\
\text{E2}\times(3): \quad &+ \quad 9x + 6y = \quad 39
\end{align*}
\]
We adjust the \( y \)'s into opposite signs then add to eliminate them.
\[
\begin{align*}
x \quad &= \quad 3 \quad \text{So a hamburger is 3$}.
\end{align*}
\]

Systems of Linear Equations
To find \( y \), set 3 for \( x \) in E2: \( 3x + 2y = 13 \)
and get \( 3(3) + 2y = 13 \)
\[
\begin{align*}
9 + 2y &= 13 \\
2y &= 4 \\
y &= 2
\end{align*}
\]
Hence the solution is \((3, 2)\).
Therefore a hamburger cost $3 and a salad cost $2.
The above method is called the elimination (addition) method.
We summarize the steps below.
1. Line up the \( x \)'s & \( y \)'s to the left and the numbers to the right.
2. Select the variable \( x \) or \( y \) to eliminate.
3. Find the LCM of the terms with that variable, and convert these terms to the LCM (by multiplication).
4. Add or subtract the adjusted equations and solve the resulting equation.
5. Substitute the answer back into any equation to solve for the other variable.
Example C. Solve \( \begin{align*} 5x + 4y &= 2 \quad \text{E1} \\ 2x - 3y &= -13 \quad \text{E2} \end{align*} \)

Eliminate the \( y \). The LCM of the \( y \)-terms is 12y.

Multiply E1 by 3 and E2 by 4.

\[ 3 \times \text{E1:} \quad 15x + 12y = 6 \]

\[ 4 \times \text{E2:} \quad 8x - 12y = -52 \]

Add: \[ 23x + 0 = -46 \rightarrow x = -2 \]

Set -2 for \( x \) in E1 and get \[ \begin{align*} 5(-2) + 4y &= 2 \\ -10 + 4y &= 2 \\ 4y &= 12 \\ y &= 3 \end{align*} \]

Hence the solution is \((-2, 3)\).

In all the above examples, we obtain exactly one solution in each case. However, there are two other possibilities.

The system might not have any solution or the system might have infinite many solutions.

---

**Systems of Linear Equations**

Two Special Cases:

I. **Contradictory (Inconsistent) Systems**

Example D.

\[ \begin{align*} x + y &= 2 \quad \text{(E1)} \\ x + y &= 3 \quad \text{(E2)} \end{align*} \]

Remove the \( x \)-terms by subtracting the equations.

\[ \begin{align*} \text{E1} - \text{E2:} \quad &x + y = 2 \\ \quad &- \) \ x + y = 3 \\ \quad &0 = -1 \end{align*} \]

This is impossible! Such systems are said to be **contradictory** or **inconsistent**.

These system don't have solution.
Systems of Linear Equations

II. Dependent Systems

Example E. \[
\begin{align*}
\begin{cases}
2x + y &= 3 \quad \text{(E1)} \\
2x + 2y &= 6 \quad \text{(E2)}
\end{cases}
\end{align*}
\]
Remove the x-terms by multiplying E1 by 2 then subtract E2.

\[
\begin{align*}
2^*\text{E1} - \text{E2} : \\
2x + 2y &= 6 \\
\underline{-} 2x + 2y &= 6 \\
0 &= 0
\end{align*}
\]

This means the equations are actually the same and it has infinitely many solutions such as (3, 0), (2, 1), (1, 2) etc... Such systems are called dependent or redundant systems.

Finally, as before with linear equations, clear the denominators of fractional equations first. For example, change

\[
\begin{align*}
\begin{cases}
\left(\frac{3}{2}x - \frac{2}{3}y = 3\right)^*6 \\
\left(\frac{1}{2}x - \frac{1}{4}y = -1\right)^*4
\end{cases}
\end{align*}
\]

then solve.

\[
\begin{align*}
\begin{cases}
9x - 4y &= 18 \\
2x - y &= -4
\end{cases}
\end{align*}
\]

Exercise. A. Select the solution of the system.

1. a. (0, 3)  b. (1, 2)  c. (2, 1)  d. (3, 0) \[
\begin{align*}
\begin{cases}
x + y &= 3 \\
2x + y &= 3
\end{cases}
\end{align*}
\]

2. a. (0, 3)  b. (1, 2)  c. (2, 1)  d. (3, 0) \[
\begin{align*}
\begin{cases}
x + y &= 3 \\
2x + y &= 4
\end{cases}
\end{align*}
\]

3. a. (0, 3)  b. (1, 2)  c. (2, 1)  d. (3, 0) \[
\begin{align*}
\begin{cases}
x + y &= 3 \\
2x + y &= 5
\end{cases}
\end{align*}
\]

4. a. (0, 3)  b. (1, 2)  c. (2, 1)  d. (3, 0) \[
\begin{align*}
\begin{cases}
x + y &= 3 \\
2x + y &= 6
\end{cases}
\end{align*}
\]

B. Solve the following systems.

5. \[
\begin{align*}
\begin{cases}
y &= 3 \\
2x + y &= 3
\end{cases}
\end{align*}
\]

6. \[
\begin{align*}
\begin{cases}
x &= 1 \\
2x + y &= 4
\end{cases}
\end{align*}
\]

7. \[
\begin{align*}
\begin{cases}
y &= 1 \\
2x + y &= 5
\end{cases}
\end{align*}
\]

8. \[
\begin{align*}
\begin{cases}
x &= 2 \\
2x + y &= 6
\end{cases}
\end{align*}
\]

C. Add or subtract vertically. (These are warm ups)

9. \[
\begin{align*}
\begin{cases}
8x + 10y &= -99 \\
-8y - 9y &= -9 \\
-3 &= 9 \\
+5 &= -5x
\end{cases}
\end{align*}
\]

10. \[
\begin{align*}
\begin{cases}
8x - 9y &= -9 \\
-3 &= 9 \\
-4y - 9y &= -5y \\
-5x &= -5x
\end{cases}
\end{align*}
\]

11. \[
\begin{align*}
\begin{cases}
-8y &= -9 \\
-3 &= 9 \\
-4y &= -5y \\
-5x &= -5x
\end{cases}
\end{align*}
\]

12. \[
\begin{align*}
\begin{cases}
-8y &= -9 \\
-3 &= 9 \\
-4y &= -5y \\
-5x &= -5x
\end{cases}
\end{align*}
\]

13. \[
\begin{align*}
\begin{cases}
-8y &= -9 \\
-3 &= 9 \\
-4y &= -5y \\
-5x &= -5x
\end{cases}
\end{align*}
\]

14. \[
\begin{align*}
\begin{cases}
-8y &= -9 \\
-3 &= 9 \\
-4y &= -5y \\
-5x &= -5x
\end{cases}
\end{align*}
\]
Systems of Linear Equations

D. Solve the following system by the elimination method.

15. \[
\begin{align*}
    x + y &= 3 \\
    2x + y &= 4
\end{align*}
\]

16. \[
\begin{align*}
    x + y &= 3 \\
    2x - y &= 6
\end{align*}
\]

17. \[
\begin{align*}
    x + 2y &= 3 \\
    2x - y &= 6
\end{align*}
\]

18. \[
\begin{align*}
    -x + 2y &= -12 \\
    2x + y &= 4
\end{align*}
\]

19. \[
\begin{align*}
    3x + 4y &= 3 \\
    x - 2y &= 6
\end{align*}
\]

20. \[
\begin{align*}
    x + 3y &= 3 \\
    2x - 9y &= -4
\end{align*}
\]

21. \[
\begin{align*}
    -3x + 2y &= -1 \\
    2x + 3y &= 5
\end{align*}
\]

22. \[
\begin{align*}
    2x + 3y &= -1 \\
    3x + 4y &= 2
\end{align*}
\]

23. \[
\begin{align*}
    4x - 3y &= 3 \\
    3x - 2y &= -4
\end{align*}
\]

24. \[
\begin{align*}
    5x + 3y &= 2 \\
    2x + 4y &= -2
\end{align*}
\]

25. \[
\begin{align*}
    3x + 4y &= -10 \\
    -5x + 3y &= 7
\end{align*}
\]

26. \[
\begin{align*}
    -4x + 9y &= 1 \\
    5x - 2y &= 8
\end{align*}
\]

27. \[
\begin{align*}
    \frac{3}{2}x - \frac{2}{3}y &= 3 \\
    \frac{1}{2}x - \frac{1}{4}y &= -1
\end{align*}
\]

28. \[
\begin{align*}
    \frac{1}{2}x + \frac{1}{5}y &= 1 \\
    \frac{3}{4}x - \frac{1}{6}y &= -1
\end{align*}
\]

E. Which system is inconsistent and which is dependent?

29. \[
\begin{align*}
    x + 3y &= 4 \\
    2x + 6y &= 8
\end{align*}
\]

30. \[
\begin{align*}
    2x - y &= 2 \\
    8x - 4y &= 6
\end{align*}
\]
3–4 Systems of Linear Equations II

Example A. At Pizza Grande, one coupon \( C \) may be exchanged for three slices of pizza \( (P) \) and five donuts \( (D) \). We have 5 slices of pizzas, 3 donuts and 2 coupons. What do we have after exchanging the 2 coupons?

The coupon value of \( C \) may be recorded as \( C = 5P + 3D \). 5 slices of pizzas, 3 donuts and 2 coupons is \( 5P + 3D + 2C \). Exchanging the 2 coupons for pizzas and donuts, we would have:

\[
\begin{align*}
5P + 3D + 2C &= 5P + 3D + 2(5P + 3D) \\
&= 5P + 3D + 10P + 6D \\
&= 15P + 9D
\end{align*}
\]

or 15 slices of pizzas and 9 donuts.

In math, the phrase "substitute (the expression) back into ...“ means to do the exchange, using the given coupon–expression, in the targeted equations, or expressions mentioned.

Systems of Linear Equations II

There are two other methods to solve system of equations.

**Substitution Method**

In substitution method, we solve for one of the variables in terms of the other, then substitute the result into the other equation.

Example B. Solve \( \begin{cases} 2x + y = 7 \cdots \text{E1} \\ x + y = 5 \cdots \text{E2} \end{cases} \)

by the substitution method.

From E2, \( x + y = 5 \), we get \( x = 5 - y \). From this, we may exchange \( x = 5 - y \) so we get 2(5 – y) + y = 7

10 – 2y + y = 7

10 – y = 7 \( \rightarrow \) \( y = 3 \)

Put \( y = 3 \) back to E2 to find \( x \), we get

\( x + 3 = 5 \rightarrow x = 2 \)

Therefore the solution is \( (2, 3) \).
Systems of Linear Equations II

We use the substitution method when it's easy to solve for one of the variable in terms of the other. Specifically, it is easy to solve for a variable when an equation contains a single x or single y.

Example C. Solve \{ \begin{align*}
2x - y &= 7 \quad \text{--- E1} \\
3x + 2y &= 7 \quad \text{--- E2}
\end{align*}
\}

by the substitution method.

By inspection, we see that it's easy to solve for the y using E1. From E1, \(2x - y = 7\), we get \(2x - 7 = y\). Substitute y by \((2x - 7)\) in E2 and get

\[
\begin{align*}
3x + 2(2x - 7) &= 7 \\
3x + 4x - 14 &= 7 \\
7x &= 21 \\
x &= 3
\end{align*}
\]

To find \(y\), use the substitution equation, set \(x = 3\) in \(y = 2x - 7\)

\[
y = 2(3) - 7 = -1
\]

Hence the solution is \((3, -1)\).

Systems of Linear Equations II

Graphing Method

Given a system of linear equations the graph of each equation is a straight line. The solution of the system is the intersection point \((x, y)\) of the these lines. Thus we may find the solution by graphing the lines and locate the point of intersection graphically. In general, we don't not use the graphing method because it is not easy to do it accurately.

Example D. Solve \{ \begin{align*}
2x + y &= 7 \quad \text{--- E1} \\
x + y &= 5 \quad \text{--- E2}
\end{align*}
\}

Use intercept method.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>(\frac{7}{2})</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

\(2x + y = 7\) \quad \(x + y = 5\)

(Check another point.)

The intersection \((2, 3)\) is the solution.
Systems of Linear Equations II

Graphing of inconsistent systems are parallel lines which do not intersect, hence there is no solution.

Example E. Solve \( \begin{cases} x + y = 7 & \text{E1} \\ x + y = 5 & \text{E2} \end{cases} \) graphically.

\[
\begin{array}{c|c}
 x & y \\
\hline
0 & 7 \\
7 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
 x & y \\
\hline
0 & 5 \\
5 & 0 \\
\end{array}
\]

No intersection. No solution

Systems of Linear Equations II

Graphs of dependent systems are identical lines, hence every point on the line is a solution.

Example F. Solve \( \begin{cases} x + y = 5 & \text{E1} \\ 2x + 2y = 10 & \text{E2} \end{cases} \) graphically.

\[
\begin{array}{c|c}
 x & y \\
\hline
0 & 5 \\
5 & 0 \\
\end{array}
\]

2x + 2y = 10 is the same as x + y = 5

Every point is a solution, e.g., (0, 5), (2, 3), (5, 0)
Systems of Linear Equations II

Exercise. Solve by the substitution method.

1. \[
\begin{cases}
y = 3 - x \\
2x + y = 4
\end{cases}
\]

2. \[
\begin{cases}
x + y = 3 \\
2x + 6 = y
\end{cases}
\]

3. \[
\begin{cases}
x = 3 - y \\
2x - y = 6
\end{cases}
\]

4. \[
\begin{cases}
-x + 2y = -12 \\
y = 4 - 2x
\end{cases}
\]

5. \[
\begin{cases}
3x + 4y = 3 \\
x = 6 + 2y
\end{cases}
\]

6. \[
\begin{cases}
x = 3 - 3y \\
2x - 9y = -4
\end{cases}
\]

Problem 7, 8, 9: Solve problem 1, 2, and 3 by graphing.

10. Graph the inconsistent system

\[
\begin{cases}
2x - y = 2 \\
8x - 4y = 6
\end{cases}
\]

11. Graph the dependent system

\[
\begin{cases}
x + 3y = 4 \\
2x + 6y = 8
\end{cases}
\]
3–5 Linear Word-Problems II

Following are key-words translated into mathematic operations.

+-: add, sum, plus, total, combine, increased by, # more than ..
-=: subtract, difference, minus, decreased by, # less than ..
*: multiply, product, times, “fractions or %” of the amount ..
/: divide, quotient, shared equally, ratio ..

Twice = Double = 2*(amount)
Square = (amount)²

In chapter 2, we solved problems with a single variable. Many of those problems may be solved using two variables and a system of equations. The advantage of using two variables instead of one is that it’s easier to construct the equations to form the system. To do this:
I. identify the unknown quantities and named them as x and y
II. usually there are two numerical relations given, translate each numerical relation into an equation to make a system
III. solve the system

Linear Word-Problems II

Example A. A 30-foot rope is cut into two pieces.
The longer one is 6 feet less than twice of the short piece.
How long is each piece?
Let L = length of the long piece
   S = length of the short piece
Then \[ \begin{align*}
L &+ S = 30 \quad \text{--- E1} \\
L &= 2S - 6 \quad \text{--- E2}
\end{align*} \]
Use the substitution method.
Substitute \( L = (2S - 6) \) into E1, we get
\[
(2S - 6) + S = 30
\]
\[
2S - 6 + S = 30
\]
\[
3S = 36
\]
\[
S = 36/3 = 12
\]
Put \( y = 12 \) into E2, we get \( x = 2(12) - 6 = 18 \).
Hence the rope is cut into 18 and 12 feet.
Linear Word-Problems

Making Tables
If the given information are the same types of
for multiple entities, organize the information into a table.
Example B. Put the following information into a table.

a. Maria's grocery list:
   6 apples, 4 bananas, 3 cakes.
   Don's grocery list:
   10 cakes, 2 apples, 6 bananas.

\[
\begin{array}{|c|c|c|}
\hline
& \text{Apple} & \text{Banana} \\
\hline
\text{Maria} & 6 & 4 \\
\text{Don} & 2 & 6 \\
\hline
\end{array}
\]
\[
\begin{array}{|c|c|c|}
\hline
& \text{Cake} \\
\hline
\text{Maria} & 3 \\
\hline
\end{array}
\]

   Mary bought 8lb of beets and 15lb of carrots.
   Don bought 12lb of carrots and 9lb of beets.

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{Beet-cost ($5/lb)} & \text{Carrot-cost ($3/lb)} & \text{Total} \\
\hline
\text{Maria} & (8 \times 5) & 40 & (15 \times 3) & 45 & 85 \\
\text{Don} & (9 \times 5) & 45 & (12 \times 3) & 36 & 81 \\
\hline
\end{array}
\]

Total: 166$

Linear Word-Problems II

Example C. Joe ordered four hamburgers and three fries.
It cost $18. Mary ordered three hamburgers and two fries and
it cost $13. Find the price of each item.
Let \( x \) = price of a hamburger, \( y \) = price of an order of fries.
The system is \[
\begin{align*}
4x + 3y &= 18 \quad \text{E1} \\
3x + 2y &= 13 \quad \text{E2}
\end{align*}
\]
Use the elimination method. LCM of 3y and 2y is 6y.
Multiply E1 by 2 and E2 by 3, subtract
\[
\begin{align*}
2^*\text{E1:} & \quad 8x + 6y = 36 \\
3^*\text{E2:} & \quad 9x + 6y = 39 \\
\hline
- x & \quad = -3 \rightarrow x = 3
\end{align*}
\]
To find \( y \), put \( x = 3 \) into E2, \( 3(3) + 2y = 13 \)
\[
\begin{align*}
9 + 2y &= 13 \\
2y &= 13 - 9 \\
2y &= 4 \\
y &= 4/2 = 2.
\end{align*}
\]
So a hamburger is $3, and an order of fries is $2.
Example C. Joe ordered four hamburgers and three fries. It cost $18. Mary ordered three hamburgers and two fries and it cost $13. Find the price of each item. Let \( x = \) price of a hamburger, \( y = \) price of an order of fries.

The system is \[
\begin{align*}
4x + 3y &= 18 \quad \text{--- E1} \\
3x + 2y &= 13 \quad \text{--- E2}
\end{align*}
\]

Use the elimination method. LCM of 3y and 2y is 6y.

Multiply E1 by 2 and E2 by 3, subtract \(2^*E1: 8x + 6y = 36\) \[3^*E2: 9x + 6y = 39\]

\[-x = -3 \Rightarrow x = 3\]

To find \( y \), put \( x = 3 \) into E2, \( 3(3) + 2y = 13\)

\[
\begin{align*}
9 + 2y &= 13 \\
2y &= 13 - 9 \\
2y &= 4 \Rightarrow y = 4/2 = 2.
\end{align*}
\]

So a hamburger is $3, and an order of fries is $2.

---

For example, a boat that travels 3 mph in still water goes upstream against a 1 mph current, it’s upstream rate is 2mph. It would take 6 hours to go 12 miles. If it travels downstream, it can travel at 4 mph and it would take 3 hours to go 12 miles.

Example D. A boat can travel upstream for 24 miles in 6 hours and the same distance downstream in 3 hours. What is the cruising speed of the boat and the current speed?

Let \( R = \) cruising speed, \( C = \) current speed.

Make a table.

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time (T)</th>
<th>Distance (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upstream</td>
<td>( R - C )</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>Downstream</td>
<td>( R + C )</td>
<td>3</td>
<td>24</td>
</tr>
</tbody>
</table>

By the formula \( R \cdot T = D \) we get two equations.
Linear Word-Problems II

{ 6(R – C) = 24 --- E1
3(R + C) = 24 --- E2

Simplify each equation, divide E1 by 6 and divide E2 by 3.

{ R – C = 4 --- E3
{ R + C = 8 --- E4
Add the equations to eliminate C

2R = 12
R = 6

Substitute R = 6 into E2,
6 + C = 8
C = 2

So, R = cruising speed = 6 mph, C = current speed = 2 mph

Simple Interest Formula
Recall the simple interest of an investment is I = R \cdot P \text{ where}
I = amount of interest for one year
R = interest (in %)
P = principal

---

Linear Word-Problems II

Example E. (Mixed investments)
We have $20,000 saved in two accounts which give 5% and 6% interest respectively. In one year, their combined interest is $1150. How much is in each account?

Make a table using the interest formula R \cdot P = I

<table>
<thead>
<tr>
<th>rate</th>
<th>principal</th>
<th>interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>6%</td>
<td>x</td>
<td>\frac{6}{100} \cdot x</td>
</tr>
<tr>
<td>5%</td>
<td>y</td>
<td>\frac{5}{100} \cdot y</td>
</tr>
<tr>
<td>total</td>
<td>20,000</td>
<td>1150</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
6x + 5y &= 115000 \quad \text{--- E3}
\end{align*}
\]

Multiply E2 by 100 to clear the denominator.

\[
\begin{align*}
\left( \frac{6}{100} \cdot x + \frac{5}{100} \cdot y \right) \cdot 100 &= 115000 \\
6x + 5y &= 115000 \quad \text{--- E3}
\end{align*}
\]
Linear Word-Problems II

\[
\begin{aligned}
\begin{cases}
  x + y &= 20,000 \text{---E1} \\
  6x + 5y &= 115000 \text{---E3}
\end{cases}
\end{aligned}
\]

To eliminate \( y \), multiply \( E1 \) by -5 and add to \( E3 \).

\[
\begin{aligned}
  E1(-5): \quad -5x - 5y &= -100,000 \\
  6x + 5y &= 115000 \\
  \hline
  \quad x &= 15000
\end{aligned}
\]

Sub \( x = 15000 \) into \( E1 \).

\[
\begin{aligned}
  15000 + y &= 20000 \\
  \quad y &= 5000
\end{aligned}
\]

So there are $15,000 in the 6% account, 
and $5,000 in the 5% account.

Exercise. A. The problems below are from the word-problems for linear equations with one variable.

1. A and B are to share $120. A gets $30 more than B, how much does each get? (Just let \( A = x \) that Mr. A has, and \( B = y \) that Mr. B has.)

2. A and B are to share $120. A gets $30 more than twice of what B gets, how much does each get?

3. A and B are to share $120. A gets $16 less than three times of what B gets, how much does each get?

Let \( x = \) number of lb of peanuts \( \quad y = \) number of lb of cashews (Make tables)

4. Peanuts costs $2/lb, cashews costs $8/lb. How many lbs of each are needed to get 50 lbs of peanut-cashews-mixture that’s $3/lb?

5. Peanuts costs $3/lb, cashews costs $6/lb. How many lbs of each are needed to get 12 lbs of peanut-cashews-mixture that’s $4/lb?

6. We have $3000 more saved at an account that gives 5% interest than at an account that gives 4% interest. In one year, their combined interest is $600. How much is in each account?

7. We have saved at a 6% account $2000 less than twice at a 4% account. In one year, their combined interest is $1630. How much is in each account?

8. The combined saving in two accounts is $12,000. One account gives 5% interest, the other gives 4% interest. The combined interest is $570 in one year. How much is in each account?
9. Find \( x = \$ \) of cashews and \( y = \$ \) of peanuts per lb, solve for \( x \) and \( y \) with the following information.

<table>
<thead>
<tr>
<th></th>
<th>Order 1 (in lb)</th>
<th>Order 2 (in lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cashews</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Peanuts</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Cost</td>
<td>$16</td>
<td>$17</td>
</tr>
</tbody>
</table>

10. Organize this information into a table and solve for \( x \) and \( y \).

<table>
<thead>
<tr>
<th></th>
<th>Order 1 (in lb)</th>
<th>Order 2 (in lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cashews</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Peanuts</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Cost</td>
<td>$17</td>
<td>$15</td>
</tr>
</tbody>
</table>

11. Order 1 consists of 5 lb of cashews and 3 lb of peanuts and it costs $16, Order 2 consists of 7 lb of cashews and 2 lb of peanuts and it costs $18. Organize this information into a table and solve for \( x \) and \( y \).

12. Order 1 consists of 3 lb of cashews and 4 lb of peanuts and it costs $38, Order 2 consists of 2 lb of cashews and 5 lb of peanuts and it costs $30. Organize this information into a table and solve for \( x \) and \( y \).

Note that 9–12 are the same as the hamburger-fries problem.

---

**Linear Word-Problems**

C. Select the \( x \) and \( y \), make a table, then solve.

13. We have $3000 more saved in an account that gives 6% interest than in an account that gives 4% interest. In one year, their combined interest is $600. How much is in each account?

14. We have saved in a 6% account $2000 less than twice the amount in a 4% account. In one year, their combined interest is $1600. How much is in each account?

15. The combined saving in two accounts is $12,000. One account has 5% interest, the other has 4% interest. The combined interest is $570 in one year. How much is in each account?

16. The combined saving in two accounts is $15,000. One account has 6% interest, the other has 3% interest. The combined interest is $750 in one year. How much is in each account?
3–6 Slopes of Lines

The steepness of a street is measured in "grade". For example:

A Seattle trolleybus climbing an 18%-grade street (Wikipedia)

The **18%-grade** means the ratio of 18 to 100 as shown here:

```
100 ft  18 ft
```

The steepness of a roof is measured in "pitch". For example:

Here is outline of a roof with a pitch of 4:12 or 1/3.

In mathematics, these measurements are called "slopes".

---

Slopes of Lines

The **slope** of a line is a number. The **slope** of a line measures the amount of tilt, (inclination, steepness) of the line against the x-axis.

Steep lines have **slopes** with large absolute value.

Gradual lines have **slopes** with small absolute value

**Definition of Slope**

Notation: The Greek capital letter Δ (delta) in general means "the difference" in mathematics.

Δy means the difference in the values of y's, Δx means the difference the values of x's.

Example A. Let \( y_1 = -2, \ y_2 = 5 \),
then \( \Delta y = y_2 - y_1 = 5 - (-2) = 7 \)

Let \( x_1 = 7, \ x_2 = -4 \),
then \( \Delta x = x_2 - x_1 = -4 - 7 = -11 \)
Slopes of Lines

Definition of Slope
Let \((x_1, y_1)\) and \((x_2, y_2)\) be two points on a line, then the slope \(m\) of the line is

\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]

Geometry of Slope
\(\Delta y = y_2 - y_1\) = the difference in the heights of the points.
\(\Delta x = x_2 - x_1\) = the difference in the runs of the points.

Therefore \(m = \frac{\Delta y}{\Delta x}\) is the ratio of the “rise” to the “run”.

\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} \]

Note that \(\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}\)

so the numbering of the two points is not relevant—we get the same answer.

---

Example B. Find the slope of the line that passes through \((3, -2)\) and \((-2, 8)\). Draw the line.
It’s easier to find \(\Delta x\) and \(\Delta y\) vertically.

\((-2, 8)\)
\((3, -2)\)

\[ \Delta x = -5, \quad \Delta y = 10 \]

Hence the slope is
\[ m = \frac{\Delta y}{\Delta x} = \frac{10}{-5} = -2 \]

Example C. Find the slope of the line that passes through \((3, 5)\) and \((-2, 5)\).
Draw the line.

\((-2, 5)\)
\((3, 5)\)

\[ \Delta x = -5, \quad \Delta y = 0 \]

So the slope is
\[ m = \frac{\Delta y}{\Delta x} = \frac{0}{-5} = 0 \]
Slopes of Lines

As shown in example C, the slope of a horizontal line is 0, i.e. it’s “tilt” is 0.

Example D. Find the slope of the line that passes through (3, 2) and (3, 5). Draw the line.

\[(3, 5) \quad \rightarrow \quad (3, 2)\]

\[\Delta x = 0, \quad \Delta y = 3\]

So the slope

\[m = \frac{\Delta y}{\Delta x} = \frac{3}{0}\]

is undefined!

Hence the slope \(m\) of a horizontal line is \(m = 0\).

Hence the slope \(m\) of a vertical line is undefined (UDF).

Hence the slope \(m\) of a tilted line is a non-zero number.

Summary of Slope

The slope of the line that passes through \((x_1, y_1)\) and \((x_2, y_2)\) is

\[m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}\]

Tilted line
Slope = \(-2 \neq 0\)

Horizontal line
Slope = 0

Vertical line
Slope is UDF.
Slopes of Lines

Exercise A.
Select two points and estimate the slope of each line.

1. [Graph showing two points]
2. [Graph showing two points]
3. [Graph showing two points]
4. [Graph showing two points]
5. [Graph showing two points]
6. [Graph showing two points]
7. [Graph showing two points]
8. [Graph showing two points]

Slopes of Lines

Exercise B. Draw and find the slope of the line that passes through the given two points. Identify the vertical line and the horizontal lines by inspection first.

9. (0, −1), (−2, 1)
10. (1, −2), (−2, 0)
11. (1, −2), (−2, −1)
12. (3, −1), (3, 1)
13. (1, −2), (−2, 3)
14. (2, −1), (3, −1)
15. (4, −2), (−3, 1)
16. (4, −2), (4, 0)
17. (7, −2), (−2, −6)
18. (3/2, −1), (3/2, 1)
19. (3/2, −1), (1, −3/2)
20. (−5/2, −1/2), (1/2, 1)
21. (3/2, 1/3), (1/3, 1/3)
22. (−2/3, −1/4), (1/2, 2/3)
23. (3/4, −1/3), (1/3, 3/2)
3–7 More on Slopes

Definition of Slope
Let \((x_1, y_1)\) and \((x_2, y_2)\) be two points on a line, then the slope \(m\) of the line is
\[
m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

Geometry of Slope
\(\Delta y = y_2 - y_1\) is the difference in the heights of the points.
\(\Delta x = x_2 - x_1\) is the difference in the runs of the points.

Therefore \(m = \frac{\Delta y}{\Delta x}\) is the ratio of the "rise" to the "run".

More on Slopes
Example A. Find the slope of each of the following lines.

<table>
<thead>
<tr>
<th>Two points are</th>
<th>((-3, 1), (4, 1))</th>
<th>((-2, -4), (2, 3))</th>
<th>((-1, 3), (6, 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta y = 1 - 1 = 0)</td>
<td>(\Delta y = 3 - (-4) = 7)</td>
<td>(\Delta y = 3 - 3 = 0)</td>
<td></td>
</tr>
<tr>
<td>(\Delta x = 4 - (-3) = 7)</td>
<td>(\Delta x = 2 - (-2) = 4)</td>
<td>(\Delta x = 6 - (-1) = 7)</td>
<td></td>
</tr>
<tr>
<td>(m = \frac{\Delta y}{\Delta x} = \frac{0}{7} = 0)</td>
<td>(m = \frac{\Delta y}{\Delta x} = \frac{7}{4})</td>
<td>(m = \frac{\Delta y}{\Delta x} = \frac{7}{0} \text{ (UDF)})</td>
<td></td>
</tr>
</tbody>
</table>

Horizontal line
Slope = 0

Tilted line
Slope = 0

Vertical line
Slope is UDF
More on Slopes

Lines that go through the quadrants I and III have positive slopes.

Lines that go through the quadrants II and IV have negative slopes.

The formula for slopes requires geometric information, i.e., the positions of two points on the line. However, if a line is given by its equation instead, we may determine the slope from the equation directly.

---

More on Slopes

Given a linear equation in x and y, solve for the variable y if possible. We get \( y = mx + b \), the number \( m \) is the slope and \( b \) is the y-intercept. This is called the slope intercept form and this can be done only if the y-term is present.

Example B. Write the equations into the slope intercept form, list the slopes, the y-intercepts and draw the lines.

a. \( 3x = -2y + 6 \) solve for y

\[
2y = -3x + 6
\]

\[
y = \frac{-3}{2} x + 3
\]

Hence the slope \( m = -3/2 \) and the y-intercept is \((0, 3)\).

Set \( y = 0 \), we get the x-intercept \((2, 0)\). Use these points to draw the line.
More on Slopes

b. \(0 = -2y + 6\) solve for \(y\)
\[
2y = 6
\]
\[
y = 3
\]
\[
y = 0x + 3
\]
Hence the slope \(m\) is 0.
The \(y\)-intercept is \((0, 3)\).
There is no \(x\)-intercept.

c. \(3x = 6\)
The variable \(y\) can't be isolated because there is no \(y\).
Hence the slope is undefined and this is a vertical line.
Solve for \(x\)
\[
3x = 6 \rightarrow x = 2.
\]
This is the vertical line \(x = 2\).

More on Slopes

Two Facts About Slopes

I. Parallel lines have the same slope.
II. Slopes of perpendicular lines are the negative reciprocal of each other.

Example C.
a. The line \(L\) is parallel to \(4x - 2y = 5\), what is the slope of \(L\)?

Solve for \(y\) for \(4x - 2y = 5\)
\[
4x - 5 = 2y
\]
\[
2x - 5/2 = y
\]
So the slope of \(4x - 2y = 5\) is 2.
Since \(L\) is parallel to it, so \(L\) has slope 2 also.

b. What is the slope of \(L\) if \(L\) is perpendicular to \(3x = 2y + 4\)?

Solve for \(y\) to find the slope of \(3x - 4 = 2y\)
\[
\frac{3}{2}x - 2 = y
\]
Hence the slope of \(3x = 2y + 4\) is \(-\frac{2}{3}\).
So \(L\) has slope \(-2/3\) since \(L\) is perpendicular to it.
Summary on Slopes

Geometry of Slope
The slope of tilted lines are nonzero.
Lines with positive slopes connect quadrants I and III.
Lines with negative slopes connect quadrants II and IV.
Lines that have slopes with large absolute values are steep.
The slope of a horizontal line is 0.
A vertical lines does not have slope or that it's UDF.
Parallel lines have the same slopes.
Perpendicular lines have the negative reciprocal slopes of each other.

How to Find Slopes
I. If two points on the line are given, use the slope formula
\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{rise} \]
\[ = \text{run} \]

II. If the equation of the line is given, solve for the y and get
slope intercept form \( y = mx + b \), then the number \( m \) is the slope.

More on Slopes
Exercise A. Identify the vertical and the horizontal lines by inspection first. Find their slopes or if it's undefined, state so.
Fine the slopes of the other ones by solving for the \( y \).

1. \( x - y = 3 \) 2. \( 2x = 6 \) 3. \( -y - 7 = 0 \)
4. \( 0 = 8 - 2x \) 5. \( y = -x + 4 \) 6. \( 2x/3 - 3 = 6/5 \)
7. \( 2x = 6 - 2y \) 8. \( 4y/5 - 12 = 3x/4 \) 9. \( 2x + 3y = 3 \)
10. \( -6 = 3x - 2y \) 11. \( 3x + 2 = 4y + 3x \) 12. \( 5x/4 + 2y/3 = 2 \)

Exercise B.
13–18. Select two points and estimate the slope of each line.
More on Slopes

Exercise C. Draw and find the slope of the line that passes through the given two points. Identify the vertical line and the horizontal lines by inspection first.

19. \((0, -1), (-2, 1)\)  
20. \((1, -2), (-2, 0)\)  
21. \((1, -2), (-2, -1)\)

22. \((3, -1), (3, 1)\)  
23. \((1, -2), (-2, 3)\)  
24. \((2, -1), (3, -1)\)

25. \((4, -2), (-3, 1)\)  
26. \((4, -2), (4, 0)\)  
27. \((7, -2), (-2, -6)\)

28. \((3/2, -1), (3/2, 1)\)  
29. \((3/2, -1), (1, -3/2)\)

30. \((-5/2, -1/2), (1/2, 1)\)  
31. \((3/2, 1/3), (1/3, 1/3)\)

32. \((-2/3, -1/4), (1/2, 2/3)\)  
33. \((3/4, -1/3), (1/3, 3/2)\)

More on Slopes

Exercise D.

34. Identify which lines are parallel and which one are perpendicular.

A. The line that passes through \((0, 1), (1, -2)\)

B.  

C.  

D. \(2x - 4y = 1\)

E. The line that’s perpendicular to \(3y = x\)

F. The line with the \(x\)–intercept at 3 and \(y\)–intercept at 6.

Find the slope, if possible of each of the following lines.

35. The line passes with the \(x\)–intercept at \(x = 2\), and \(y\)–intercept at \(y = -5\).
More on Slopes

Find the slope, if possible, of each of the following lines

36. The equation of the line is $3x = -5y + 7$
37. The equation of the line is $0 = -5y + 7$
38. The equation of the line is $3x = 7$
39. The line is parallel to $2y = 5 - 6x$
40. The line is perpendicular to $2y = 5 - 6x$
41. The line is parallel to the line in problem 30.
42. The line is perpendicular to the line in problem 31.
43. The line is parallel to the line in problem 33.
44. The line is perpendicular to the line in problem 33.
### 3–8 Equations of Lines

Given enough information about a line, we can reconstruct an equation of the line. We separate them into two cases.

**Case I. Horizontal and Vertical Lines (The Special Case)**

- **The slope of horizontal lines is 0.** Hence the equations of horizontal lines are \( y = c \).
- **The slope of vertical lines is undefined,** i.e., there is no "y" in the equation. So the equations of vertical lines are \( x = c \).

Horizontal lines have slope 0. Slope of vertical line is undefined.

### Equations of Lines

**Example A.**

a. A line passes through \((3, -1)\), \((3, -3)\). Draw. Find its equation.

It's a vertical line. So the equation is \( x = c \) for some \( c \).
Since \((3, -1)\) is on the line so the equation must be \( x = 3 \).

b. A line passes through \((3, -1)\) and it's parallel to the \( x \)-axis. Draw. Find its equation.

Because it's parallel to the \( x \)-axis, it must be a horizontal line. So the equation is \( y = c \) for some \( c \). Since \((3, -1)\) is on the line so the equation must be \( y = -1 \).
Equations of Lines

Case II. Tilted Lines (The General Case)
To find the equations of tilted lines, use the formula below. It gives the slope-intercept equations directly. We need the slope and a point on the line to use this formula.

The Point Slope Formula (for composing the equations)
Given the slope $m$, and a point $(x_1, y_1)$ on the line, then

$$y = m(x - x_1) + y_1$$

is the equation of the line.

Example B. Find the equations of the following lines.

a. The line with slope -2 and y-intercept at -7.
   The slope is $-2$, the point is $(0, -7)$. Hence,
   $$y = -2(x - 0) + (-7)$$
   or $$y = -2x - 7$$

b. The line that contains $(1, -2)$ with the x-intercept at -4.
   We have two points on the line $(1, -2), (-4, 0)$ and we need the slope. Use the slope formula,
   $$m = \frac{\Delta y}{\Delta x} = \frac{0 - (-2)}{-4 - 1} = \frac{2}{-5}$$
   using the point $(-4, 0)$, plug in the Point Slope Formula
   $$y = \frac{2}{5}(x - (-4)) + 0$$
   $$y = \frac{2}{5}(x + 4)$$
   $$y = \frac{2}{5}x + \frac{8}{5}$$ (or $5y = -2x - 8$)

Recall that parallel lines have the same slope and perpendicular lines have slopes that are the negative reciprocals of each other.
Equations of Lines

c. The line that passes through \((3, -1)\) and is parallel to the line \(3y - 4x = 2\).
Our line has the same slope as the line \(3y - 4x = 2\).
To find the slope of \(3y - 4x = 2\), solve for the \(y\).
\[3y = 4x + 2\]
\[y = \frac{4}{3}x + \frac{2}{3}\]
Therefore the slope of the line \(3y - 4x = 2\) is \(\frac{4}{3}\).
So our line has slope \(\frac{4}{3}\).
By the point-slope formula, the equation is
\[y = \frac{4}{3}(x - 3) + (-1)\]
\[y = \frac{4}{3}x - 4 - 1\]

---

d. The line that has y-intercept at \(-3\) and is perpendicular to the line \(2x - 3y = 2\).
For the slope, solve \(2x - 3y = 2\)
\[-3y = -2x + 2\]
\[y = \frac{2}{3}x - \frac{2}{3}\]
Hence the slope of \(2x - 3y = 2\) is \(-\frac{2}{3}\).
Since perpendicular lines have slopes that are the negative reciprocals of each other, our slope is \(\frac{3}{2}\).
Hence the equation for our line is
\[y = \frac{-3}{2}(x - (0)) + (-3)\]
\[y = \frac{-3}{2}x - 3\]
Linear Equations and Lines

Many real-world relations between two quantities are linear. For example, the cost $y$ is a linear formula of $x$—the number of apples bought. For those relations that we don’t know whether they are linear or not, linear formulas give us the most basic “educated guesses.” The following example demonstrates that these problems are pondered by people ancient or present alike.

Example C. We live by a river that floods regularly. On a rock by the river bank there is a mark indicating the highest point the water level ever reached in the recorded time. At 12 pm on July 11, the water level is 28 inches from this mark. At 8 am on July 12 the water level is 18 inches from this mark. Let $x$ be a measurement for time, and $y$ be the distance from the water level and the mark. Find the linear equation between $x$ and $y$. At 4 pm July 12, the water level is 12 inches from the mark, is the flood easing or intensifying?

Equations of Lines

The easiest way to set the time measurement $x$ is to set $x = 0$ (hr) to the time of the first observation. Hence set $x = 0$ at 12 pm July 11. Therefore at 8 am of July 12, $x = 20$.

In particular, we are given that at $x = 0$, $y = 28$, and at $x = 20$, $y = 18$. We want the equation $y = m(x - x_1) + y_1$ of the line that contains the points $(0, 28)$ and $(20, 18)$.

The slope $m = \frac{\Delta y}{\Delta x} = \frac{28 - 18}{0 - 20} = -1/2$

Hence the linear equation is $y = -1/2(x - 0) + 28$ or that $y = -\frac{x}{2} + 28$

At 4 pm July 12, $x = 28$. According to the formula $y = -28/2 + 28 = -14 + 28 = 14$. But our actual observation, the water level is only 12 inches from the mark. Hence the flood is intensifying. The linear equation that we found is also called a trend line and it is shown below.
Linear Equations and Lines

\[ x = \text{number of hours passed since 12 pm July 11} \]
\[ y = \text{distance from the water level to the high mark} \]

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Linear Equations and Lines

Exercise A. For problems 1–8 select two points and estimate the slope, and find an equation of each line.
Linear Equations and Lines

Exercise B. Draw each line that passes through the given two points. Find the slope and an equation of the line. Identify the vertical lines and the horizontal lines by inspection first.

9. (0, −1), (−2, 1)  10. (1, −2), (−2, 0)  11. (1, −2), (−2, −1)
12. (3, −1), (3, 1)  13. (1, −2), (−2, 3)  14. (2, −1), (3, −1)
15. (4, −2), (−3, 1)  16. (4, −2), (4, 0)  17. (−2, −2), (−2, −6)
18. (3/2, −1), (3/2, 1)  19. (3/2, −1), (1, −3/2)
20. (−5/2, −1/2), (1/2, 1)  21. (3/2, 1/3), (1/3, 1/3)
22. (−1/4, −5/6), (2/3, −3/2)  23. (3/4, −1/3), (1/3, 3/2)

Exercise C. Find the equations of the following lines.

24. The line that passes through (0, 1) and has slope 3.
25. The line that passes through (−2, 1) and has slope −1/2.
26. The line that passes through (5, 2) and is parallel to y = x.
27. The line that passes through (−3, 2) and is perpendicular to −x = 2y.

Exercise D.

Find the equations of the following lines.

28. The line that passes through (0, 1), (1, −2)

29. It's perpendicular to $2x - 4y = 1$ and passes through (−2, 1)
30. It's perpendicular to $3y = x$ with x-intercept at $x = −3$.
31. It has y-intercept at $y = 3$ and is parallel to $3y + 4x = 1$.
32. It's perpendicular to the y-axis with y-intercept at 4.
33. It has y-intercept at $y = 3$ and is parallel to the x axis.
34. It's perpendicular to the x-axis containing the point (4, −3).
35. It is parallel to the y axis has x-intercept at $x = −7$.
36. It is parallel to the x axis has y-intercept at $y = 7$. 
Linear Equations and Lines

The cost $y$ of renting a tour boat consists of a base-cost plus
the number of tourists $x$. With 4 tourists the total cost is $65,
with 11 tourists the total is $86.$
39. What is the base cost and what is the charge per tourist?
40. Find the equation of $y$ in terms of $x$.
41. What is the total cost if there are 28 tourists?
The temperature $y$ of water in a glass is rising slowly.
After 4 min. the temperature is 30 °C, and after 11 min. the
temperature is up to 65 °C. Answer 42–44 assuming the
temperature is rising linearly.
42. What is the temperature at time 0 and what is the rate of
the temperature rise?
43. Find the equation of $y$ in terms of time.
44. How long will it take to bring the water to a boil at 100 °C?

Linear Equations and Lines

The cost of gas $y$ on May 3 is $3.58 and on May 9 is $4.00.$
Answer 45–47 assuming the price is rising linearly.
45. Let $x$ be the date in May, what is the rate of increase in
price in terms of $x$?
46. Find the equation of the price in term of the date $x$ in May.
47. What is the projected price on May 20?

48. In 2005, the most inexpensive tablet cost $900. In the year
2010, it was $500. Find the equation of the price $p$ in terms of
time $t$. What is the projected price for this year?
4–1 Exponents

We write “1” times the quantity “A” repeatedly N times as A\(^N\), i.e.

\[1 \times A \times A \times A \ldots \times A = A^N\]

Example A.

\[4^3 = (4)(4)(4) = 64\]
\[(xy)^2 = (xy)(xy) = x^2y^2\]
\[xy^2 = (x)(yy)\]
\[-x^2 = -(xx)\]

Rules of Exponents

Multiply–Add Rule: \(A^N A^K = A^{N+K}\)

Example B.

a. \(5^3 5^4 = (5 \times 5 \times 5)(5 \times 5 \times 5 \times 5) = 5^{3+4} = 5^7\)

b. \(x^5 y^7 x^4 y^8 = x^{5+4} y^{7+8} = x^9 y^{15}\)

Divide–Subtract Rule: \(\frac{A^N}{A^K} = A^{N-K}\)

Example C. \(\frac{5^6}{5^2} = \frac{(5)(5)(5)(5)(5)(5)}{(5)(5)} = 5^{6-2} = 5^4\)

Exponents

Power–Multiply Rule: \((A^N)^K = A^{NK}\)

Example D. \((3^4)^5 = (3^4)(3^4)(3^4)(3^4)(3^4) = 3^{4+4+4+4+4} = 3^{20}\)

Since \(\frac{A^1}{A^1} = 1 = A^{1-1} = A^0\), we obtain the 0-power Rule.

0-Power Rule: \(A^0 = 1\), \(A^0 \neq 0\)

Since \(\frac{1}{A^K} = \frac{A^0}{A^K} = A^{0-K} = A^{-K}\), we get the negative-power Rule.

Negative-Power Rule: \(A^{-K} = \frac{1}{A^K}\), \(A^{-K} \neq 0\)

Example D. Simplify

a. \(3^0 = 1\)

b. \(3^{-2} = \frac{1}{3^2} = \frac{1}{9}\)

c. \((\frac{2}{5})^{-1} = \frac{1}{\frac{2}{5}} = \frac{5}{2}\)

In general \((\frac{a}{b})^{-K} = (\frac{b}{a})^K\)

d. \((\frac{2}{5})^{-2} = (\frac{5}{2})^2 = \frac{25}{4}\)
Exponents

e. \(3^{-1} - 4^0 \cdot 2^{-2} = \frac{1}{3} - 1 \cdot \frac{1}{2^2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\)

Although the negative power means to reciprocate, for problems of consolidating exponents, we do not reciprocate the negative exponents. Instead we add or subtract them using the multiplication and division rules first.

Example E. Simplify \(3^{-2}x^4y^6x^8y^{-23}\)

\[\begin{align*}
3^{-2}x^4y^6x^8y^{-23} &= 3^{-2}x^4x^8y^6y^{-23} \\
&= \frac{1}{9}x^{4+8}y^{6+(-23)} \\
&= \frac{1}{9}x^{12}y^{-17} \\
&= \frac{1}{9x^4}y^{17} \\
&= \frac{y^{17}}{9x^4}
\end{align*}\]

Exponents

Example F. Simplify \(\frac{2^3x^3}{2^6x^{-3}}\) using the rules for exponents. Leave the answer in positive exponents only.

\[\frac{2^3x^3}{2^6x^{-3}} = 2^{3-6}x^{3-(-3)} = \frac{1}{2^3} \cdot \frac{1}{x^{-6}} = \frac{1}{8x^6}\]

Example G. Simplify \(\frac{(3x^{-2}y^3)^{-2}x^2}{3^{-5}x^{-3}(y^{-1}x^2)^3}\)

\[\begin{align*}
(3x^{-2}y^3)^{-2}x^2 &= \frac{3^{-2}x^{-4}y^{-6}x^2}{3^{-5}x^{-3}y^{-3}x^6} \\
&= \frac{3^{-2}x^2y^{-6}}{3^{-5}x^{-3}y^{-3}} = 3^{-2-(-5)}x^2y^{-6-(-3)} \\
&= 3^{3}x^2y^{-3} = \frac{27x^3}{y^3}
\end{align*}\]
Exercise A. Write the numbers without the negative exponents and compute the answers.

1. \(2^{-1}\)  
2. \(-2^{-2}\)  
3. \(2^{-3}\)  
4. \((-3)^{-2}\)  
5. \(3^{-3}\)  
6. \(5^{-2}\)  
7. \(4^{-3}\)  
8. \((\frac{1}{2})^{-3}\)  
9. \((\frac{2}{3})^{-1}\)  
10. \((\frac{3}{2})^{-2}\)  
11. \(2^{-1} \cdot 3^{-2}\)  
12. \(2^{-2} \cdot 3^{-1}\)  
13. \(2^{-2} \cdot 5^{-1} \cdot 3^{-1}\)  
14. \(3^{2} \cdot 6^{-1} - 6 \cdot 2^{-3}\)  
15. \(2^{-2} \cdot 3^{-1} + 6^{0} \cdot 2^{-1}\)

B. Combine the exponents. Leave the answers in positive exponents—do not reciprocate the negative exponents until the final step.

16. \(x^{3}y^{5}\)  
17. \(x^{-3}y^{6}\)  
18. \(x^{5}y^{-6}\)  
19. \(x^{-3}y^{-6}\)  
20. \(x^{2}y^{3}x^{4}y^{-4}\)  
21. \(y^{-3}x^{2}y^{-4}x^{4}\)  
22. \(2^{3}x^{3}y^{2}x^{2}y^{2}\)  
23. \(3y^{2} \cdot 5y^{3}y^{-6}\)  
24. \(4x^{2}y^{2} \cdot 3y^{3}x^{-4}y^{-1}\)  
25. \(x^{2}(x^{3})^{5}\)  
26. \((x^{-3})^{-6}x^{-4}\)  
27. \(x^{4}(x^{3}y^{-3})^{-3}y^{-6}\)

28. \(\frac{x^{6}}{x^{-3}}\)  
29. \(\frac{x^{8}}{x^{3}}\)  
30. \(\frac{x^{-8}}{x^{3}}\)  
31. \(\frac{y^{5}x^{-8}}{x^{3}y^{-3}}\)

32. \(\frac{x^{2}y^{3}y^{-8}}{y^{-3}x^{2}y^{2}}\)  
33. \(\frac{2^{3}y^{6}y^{-8}}{2^{-5}y^{3}x^{4}}\)  
34. \(\frac{3^{-2}y^{2}x^{4}}{2^{-3}x^{2}y^{2}}\)  
35. \(\frac{4^{-1}(x^{4}y^{-3})^{-3}}{2^{-3}(y^{-3}x^{2})^{-1}}\)  
36. \(\frac{6^{-2}y^{2}(x^{4}y^{-3})^{-1}}{9^{-1}(x^{3}y^{-2})^{-1}y^{2}}\)

C. Combine the exponents as far as possible.

38. \(2^{3}x^{2}\)  
39. \(3^{x+3}x^{3}\)  
40. \(a^{x+3}a^{x+5}\)  
41. \((e^{2}x^{3})e^{-x}\)  
42. \(e^{x}e^{2x+1}e^{-x}\)

43. \(e^{2x+1}e^{-x}\)

44. How would you make sense of \(2^{x^{7}}\)?
4–2 Scientific Notation

An important application for exponents is the usage of the powers of 10 in calculation of very large or very small numbers.

Powers of 10:

\[
\begin{align*}
10^3 &= 1000 \\
10^2 &= 100 \\
10^1 &= 10 \\
10^0 &= 1 \\
10^{-1} &= 0.1 \\
10^{-2} &= 0.01 \\
10^{-3} &= 0.001 \\
10^{-4} &= 0.0001
\end{align*}
\]

(starting with) pack 0’s to the right for positive exponents so they get larger
(pack 0’s to the left for negative exponents so they get smaller)

If \( r \) is a number then \( r \times 10^k = \text{shifting the decimal point of } r \), if \( k \) is positive (+), shift the point right, if \( k \) is negative (−), shift left.

In particular, every number \( x \) can be written in the form \( r \times 10^k \) with \( 1 \leq r < 10 \).

This form is called the \textit{scientific notation} of \( x \).

For example, \( 2500 = 2.5 \times 10^3 \) and that \( 0.25 = 2.5 \times 10^{-1} \).

---

Scientific Notation

Let’s change a number in scientific notation \( r \times 10^k \) back to the standard form by moving the decimal point of \( r \) according to \( N \).

i. If \( N \) is positive, move the decimal point of \( r \) to the right, i.e. make \( r \) into a larger number.

ii. If \( N \) is negative, move the decimal point of \( r \) to the left, i.e. make \( r \) into a smaller number.

Example A. Write the following numbers in the standard form.

a. \( 2.3 \times 10^{-4} = 0.00023 \) = 0.00023

Move right 4 places.

b. \( 2.3 \times 10^{-3} = 0.00023 \) = 0.00023

Move left 3 places

To represent a number \( x \) with scientific notation as \( r \times 10^N \), first identify the \( r \) using \( x \), then multiply \( r \) by \( 10^N \) to adjust the decimal point of \( r \) to get back the \( x \). To find \( N \), we count.
Scientific Notation
To express a given number \( x \) with scientific notation as \( r \times 10^n \), move the decimal point of \( x \) to the back of the first nonzero digit, this is \( r \).

i. If the point moved left \( N \) spaces so the \( r \) is smaller than \( x \), then use positive exponent \( N \) to compensate for the change.

ii. If the point moved to the right \( N \) spaces so \( r \) is more than \( x \), then use negative exponent \( N \) to compensate for the change.

Example B. Write the following numbers in scientific notation.

a. \( 12300. = 12300. \times 10^{-4} \) 
   Move left 4 places.

b. \( 0.00123 = 0.00123 \times 10^{-3} \) 
   Move right 3 places

Scientific notation simplifies complicated calculation of very large and very small numbers.

Scientific Notation
Example C. Calculate. Give the answer in both scientific notation and the standard notation.

a. \( (1.2 \times 10^3) \times (1.3 \times 10^{-12}) \) 
   \( = 1.2 \times 1.3 \times 10^3 \times 10^{-12} \) 
   \( = 1.56 \times 10^{3-12} \) 
   \( = 1.56 \times 10^{-9} \) 
   \( = 0.000156 \)

b. \( \frac{6.3 \times 10^2}{2.1 \times 10^{-16}} \) 
   \( = \frac{6.3}{2.1} \times 10^{2-(-16)} \) 
   \( = 3 \times 10^{18} \) 
   \( = 300,000,000 \)
Scientific Notation

Example D. Convert each number into scientific notation. Calculate the result. Give the answer in both scientific notation and the standard notation.

\[
240,000,000 \times 0.0000025 = 0.00015
\]

\[
= \frac{2.4 \times 10^8 \times 2.5 \times 10^{-6}}{1.5 \times 10^{-3}}
\]

\[
= \frac{2.4 \times 2.5 \times 10^8 \times 10^{-6}}{1.5}
\]

\[
= \frac{2.4 \times 2.5}{1.5} \times 10^{8-(-6)-(-4)}
\]

\[
= 4 \times 10^5 = 4,000,000
\]

For calculators, the 10th portion in scientific notation is displayed as E+N or E−N where E means exponents. Hence 2.5 \times 10^{-6} is displayed as 2.5 E−6 on the calculators.

Scientific Notation

Exercise A. Convert the following numbers into standard notation. Remember that multiplying by 10positive power makes a number larger, and multiplying by 10negative power makes the number smaller.

1. \(2.41 \times 10^3\)
2. \(2.41 \times 10^{-3}\)
3. \(315 \times 10^{-2}\)
4. \(315 \times 10^2\)
5. \(0.762 \times 10^{-5}\)
6. \(0.762 \times 10^5\)
7. \(5.93 \times 10^{-6}\)
8. \(593000 \times 10^{-6}\)

B. Convert the following numbers in scientific notation into standard notation.

9. \(5.31 \times 10^3\)
10. \(2.41 \times 10^{-3}\)
11. \(3.1 \times 10^{-2}\)
12. \(9 \times 10^2\)
13. \(5.89 \times 10^6\)
14. \(7.11 \times 10^{-6}\)
15. \(5.2 \times 10^{-9}\)
16. \(7.11 \times 10^{11}\)

C. Convert the following numbers into the scientific notation.

17. 10
18. 20
19. 200
20. 2,310
21. 231,000
22. 231,000,000,000,000
23. 1
24. 0.1
25. 0.02
26. 0.0002
27. 0.00000231
28. 0.000000000000231
C. Convert the following numbers into the scientific notation.
   28. 23,100           29. 0.0231          30. 2.31
D. Calculate by converting into the scientific notation first. Put the answers in both the scientific and standard notations.
   31. 100 x 0.01    32. 100 x 0.1     33. 100 x 0.001
   34. 4,000 x 1,200,000    35. 0.000004 x 0.000012
   36. 400,000 x 0.0000012    37. 0.004 x 1,200,000
   38. 0.005 x 2,400,000   39. 0.00024 x 600,000
   40. 0.0024 x 0.00006      41. 240,000,000 x 600,000,000
   42. \[
   \frac{240 \times 0.000004}{0.00016}
   \]
   43. \[
   \frac{0.00024 \times 600,000}{0.00015 \times 0.0008}
   \]
   44. \[
   \frac{1,400,000,000 \times 6,000}{0.0000035}
   \]
   45. \[
   \frac{0.00024 \times 0.00005}{150 \times 800}
   \]
46. Google “scientific notation applications” and list one that impresses you.
4–3 Polynomial Expressions

A mathematics expression is a calculation procedure which is written using numbers, variables, and operation-symbols. The purposes of this symbolic-form are clarity and simplicity.

For example, let x represent a number, then “2 + 3x” means to “sum 2 and 3 times x”. “4x^2 – 5x” means to “subtract 5 times x from 4 times the square of x”, (3 – 2x)^2 means to “square of the difference of 3 and twice x”.

The simplest form of expression are \#x^N, where N is a non-negative integer and \# is a number, is called a monomial (one-term). For example –1, 2x, 3x^2, and –4x^3 are monomials. If N = 0 we’ve the constants, N = 1, the linear monomials \#x.

Example A. Evaluate the monomials if y = –4

a. 3y^2
   3y^2 \to 3(–4)^2 = 3(16) = 48

Polynomial Expressions

b. –3y^2 \ (y = –4)
   –3y^2 \to –3(–4)^2

c. –3y^3
   –3y^3 \to –3(–4)^3
   = – 3(–64) = 192

Polynomial Expressions

The sum of monomials are called polynomials (many-terms). These are expressions of the form, arranged in the order of powers of the x: \#x^N \pm \#x^{N-1} \pm \ldots \pm \#x^1 \pm \#

where the \#’s are numbers.

The highest exponent N is the degree of the polynomial.

For example, 4x – 7 is 1st degree (linear) and the degree of 1 – 3x^2 – m^4\theta is 40.

The expression \frac{1}{x} is not a polynomial.
Polynomial Expressions

b. \(-3y^2\)  \((y = -4)\)
   \[-3y^2 \rightarrow -3(-4)^2\]
   \[= -3(16) = -48.\]

c. \(-3y^3\)
   \[-3y^3 \rightarrow - 3(-4)^3\]
   \[= -3(-64) = 192\]

Polynomial Expressions
The sum of monomials are called **polynomials** (many-terms).
These are expressions of the form, arranged in the order of
powers of the \(x\):
\[
\#x^n \pm \#x^{n-1} \pm \ldots \pm \#x^1 \pm \#
\]
where the \(#\)'s are numbers.

**The highest exponent \(N\) is the degree of the polynomial.**
For example, \(4x - 7\) is 1\(^{st}\) degree (linear)
and the degree of \(1 - 3x^2 - 7x^{40}\) is 40.

The expression \(\frac{1}{x}\) is not a polynomial.
Polynomial Expressions

b. \(-3y^2\) (y = -4)
\[-3y^2 \rightarrow -3(-4)^2\]
\[= -3(16) = -48.\]

c. \(-3y^3\)
\[-3y^3 \rightarrow - 3(-4)^3\]
\[= - 3(-64) = 192\]

Polynomial Expressions

The sum of monomials are called *polynomials* (many-terms). These are expressions of the form, arranged in the order of powers of the x:

\[\# x^n \pm \# x^{n-1} \pm \ldots \pm \# x^1 \pm \#\]

where the #'s are numbers.

*The highest exponent \(N\) is the degree of the polynomial.*

For example, \(4x - 7\) is 1st degree (linear)
and the degree of \(1 - 3x^2 - 7x^4\) is 4.

The expression \(\frac{1}{x}\) is not a polynomial.

---

**Polynomial Expressions**

Example D. Expand and simplify.

a. \(2(3xy - 4x^2y) + 2xy - 3xy^2\)
\[= 6xy - 8x^2y + 2xy - 3xy^2\]
\[= 8xy - 8x^2y - 3xy^2\]

b. Evaluate \(8xy - 8x^2y - 3xy^2\) if \(x = 2\).
Input \(x = 2\), we get:
\[8(2)y - 8(2)^2y - 3(2)y^2\]
\[= 16y - 32y - 6y^2\]
\[= -16y - 6y^2\]

c. Evaluate \(8xy - 8x^2y - 3xy^2\) if \(x = 2\) and \(y = 3\)
We may put \(x = 2\), \(y = 3\) into the formula and do everything all over again or we may plug into \(y = 3\) into part b which is easier. We will do the easy way.
Input \(y = 3\) into \(-16y - 6y^2\)
we get:
\[-16(3) - 6(3)^2\]
\[= -48 - 54 = -102\]
Polynomial Expressions

Ex. A. Evaluate each monomials with the given values.
1. \(2x\) with \(x = 1\) and \(x = -1\)
2. \(-2x\) with \(x = 1\) and \(x = -1\)
3. \(2x^2\) with \(x = 1\) and \(x = -1\)
4. \(-2x^2\) with \(x = 1\) and \(x = -1\)
5. \(5y^3\) with \(y = 2\) and \(y = -2\)
6. \(-5y^3\) with \(y = 2\) and \(y = -2\)
7. \(5z^4\) with \(z = 2\) and \(z = -2\)
8. \(-5y^4\) with \(z = 2\) and \(z = -2\)

B. Evaluate each monomials with the given values.
9. \(2x^2 - 3x + 2\) with \(x = 1\) and \(x = -1\)
10. \(-2x^2 + 4x - 1\) with \(x = 2\) and \(x = -2\)
11. \(3x^2 - x - 2\) with \(x = 3\) and \(x = -3\)
12. \(-3x^2 - x + 2\) with \(x = 3\) and \(x = -3\)
13. \(-2x^3 - x^2 + 4\) with \(x = 2\) and \(x = -2\)
14. \(-2x^3 - 5x^2 - 5\) with \(x = 3\) and \(x = -3\)

C. Expand and simplify.
15. \(5(2x - 4) + 3(4 - 5x)\)
16. \(6(2x - 4) - 3(4 - 5x)\)
17. \(-2(3x - 8) + 3(4 - 5x)\)
18. \(-2(3x - 8) - 3(4 - 9x)\)
19. \(7(-2x - 7) - 3(4 - 3x)\)
20. \(-5(-2 - 8x) + 7(-2 - 11x)\)

---

Polynomial Expressions

21. \(x^2 - 3x + 5 + 2(-x^2 - 4x - 6)\)
22. \(x^2 - 3x + 5 - 2(-x^2 - 4x - 6)\)
23. \(2(x^2 - 3x + 5) + 5(-x^2 - 4x - 6)\)
24. \(2(x^2 - 3x + 5) - 5(-x^2 - 4x - 6)\)
25. \(-2(3x^2 - 2x + 5) + 5(-4x^2 - 4x - 3)\)
26. \(-2(3x^2 - 2x + 5) - 5(-4x^2 - 4x - 3)\)
27. \(4(3x^3 - 5x^2) - 9(6x^2 - 7x) - 5(-8x - 2)\)
28. \(-6(7x^2 + 5x - 9) - 7(-3x^2 - 2x - 7)\)
29. Simplify \(2(3xy - xy^2) - 2(2xy - xy^2)\) then evaluated it
   with \(x = -1\), afterwards evaluate it at \((-1, 2)\) for \((x, y)\)
30. Simplify \(x^2 - 2(3xy - x^2) - 2(y^2 - xy)\) then evaluated it
   with \(y = -2\), afterwards evaluate it at \((-1, -2)\) for \((x, y)\)
31. Simplify \(x^2 - 2(3xy - z^2) - 2(z^2 - x^2)\) then evaluated it
   with \(x = -1\), \(y = -2\) and \(z = 3\).
4–4 Polynomial Operations

In the last section, we introduced polynomial expressions. Given a polynomial, each monomial is called a term.

Terms with the same variable part are called like-terms. Like-terms may be combined.

For example, \(4x + 5x - 3x^2 - 5x^2 = 9x - 2x^2\).

Unlike terms may not be combined. In particular \(x^2 + x^2 = 2x^2 \neq x^4\), and that \(x^2 + x^3 = x^2 + x^3 \neq x^5\). Note that we write \(1x^n\) as \(x^n\), \(-1x^n\) as \(-x^n\).

When multiplying a number with a term, we multiply it to the coefficient. Hence, \(3(5x) = (3 \cdot 5)x = 15x\), and \(-2(-4x) = (-2)(-4)x = 8x\).

When multiplying a number to a polynomial, we may expand the result using the distributive law: \(A(B \pm C) = AB \pm AC\).

Polynomial Operations

Example A. Expand and simplify.

a. \(3(2x - 4) + 2(4 - 5x)\)
   \[= 6x - 12 + 8 - 10x\]
   \[= -4x - 4\]

b. \(-3(x^2 - 3x + 5) - 2(-x^2 - 4x - 6)\)
   \[= -3x^2 + 9x - 15 + 2x^2 + 8x + 12\]
   \[= -x^2 + 17x - 3\]

When multiply a term with another term, we multiply the coefficient with the coefficient and the variable with the variable.

Example B.

a. \((3x^2)(2x^3) = 3 \cdot 2x^2x^3 = 6x^5\)

b. \(3x^2(-4x) = 3(-4)x^2x = -12x^3\)

c. \(3x^2(2x^3 - 4x)\) distribute
   \[= 6x^5 - 12x^3\]
Polynomial Operations

To multiply two polynomials, we may multiply each term of one polynomial against other polynomial then expand and simplify.
Example C.

a. \((3x + 2)(2x - 1)\)
\[= 3x(2x - 1) + 2(2x - 1)\]
\[= 6x^2 - 3x + 4x - 2\]
\[= 6x^2 + x - 2\]
b. \((2x - 1)(2x^2 + 3x - 4)\)
\[= 2x(2x^2 + 3x - 4) - 1(2x^2 + 3x - 4)\]
\[= 4x^3 + 6x^2 - 8x - 2x^2 - 3x + 4\]
\[= 4x^3 + 4x^2 - 11x + 4\]

Note that if we did \((2x - 1)(3x + 2)\) or \((2x^2 + 3x - 4)(2x - 1)\) instead, we get the same answers. (Check this.)

**Fact. If P and Q are two polynomials then PQ \equiv QP.**
A shorter way to multiply is to bypass the 2nd step and use the general distributive law.

---

Polynomial Operations

**General Distributive Rule:**
\[(A \pm B \pm C \pm \ldots)(a \pm b \pm c \ldots)\]
\[= Aa \pm Ab \pm Ac \ldots \pm Ba \pm Bb \pm Bc \ldots \pm Ca \pmCb \pm Cc \ldots\]

**Example D. Expand**

a. \((x + 3)(x - 4)\)
\[= x^2 - 4x + 3x - 12 \quad \text{simplify}\]
\[= x^2 - x - 12\]
b. \((x - 3)(x^2 - 2x - 2)\)
\[= x^2 - 2x^2 - 2x - 3x^2 + 6x + 6\]
\[= x^2 - 5x^2 + 4x + 6\]

*We will address the division operation of polynomials later-after we understand more about the multiplication operation.*
Polynomial Operations

We include the following method for multiplying polynomials. Instead of expanding all the terms then collect like-terms, we scan the like-terms and collect them starting from the highest degree term in the answer. We demonstrate this method and double check the answer in part b.

Example E. Multiply
a. \((x - 3)(x^2 - 2x - 2)\)

The highest degree term in the product is the \(x^3\) term so we start with the \(x^3\)-term. It’s \(x^3 x^2 = 1x^5\).

Next collect all the \(x^2\) terms in the product, they are
\(-3x^2\) and \(x^2(-2x)\) which gives \(-3x^2 - 2x^2 = -5x^2\).

Next collect the \(x\)-terms and they are \(x^1(-2) + -3(-2x) = 4x\).

The last term is the number term or \(-3(-2) = 6\).

\[
(x - 3)(x^2 - 2x - 2) = x^3 - 5x^2 + 4x + 6 \quad \text{(same as before)}
\]

Polynomial Operations

Ex. A. Multiply the following monomials.
1. \(3x^2(-3x^2)\)
2. \(-3x^2(8x^6)\)
3. \(-5x^2(-3x^3)\)
4. \(-12(\frac{-5x^3}{6})\)
5. \(24\left(\frac{-5}{8}\right)x^3\)
6. \(6x^2\left(\frac{2x^5}{3}\right)\)
7. \(-15x^4\left(\frac{-2}{5}x^5\right)\)

B. Fill in the degrees of the products.
8. \(#x(#x^2 + #x + #) = #x^3 + #x^2 + #x\)
9. \(#x^2(#x^4 + #x^3 + #x^2) = #x^6 + #x^5 + #x^2\)
10. \(#x^4(#x^3 + #x^2 + #x + #) = #x^7 + #x^6 + #x^5 + #x^4\)

C. Expand and simplify.
11. \(4x(3x - 5) - 9(6x - 7)\)
12. \(-x(2x + 7) + 3(4x - 2)\)
13. \(-3x(3x + 2) - 8x(7x - 5)\)
14. \(5x(-5x + 9) + 6x(6x - 1)\)
15. \(2x(-4x + 2) - 3x(2x - 1) - 3(4x - 2)\)
16. \(-4x(-7x + 9) - 2x(2x - 5) + 9(4x + 2)\)
Polynomial Operations

D. Expand and simplify. (Use any method.)

18. \((x + 5)(x + 7)\)  
19. \((x - 5)(x + 7)\)

20. \((x + 5)(x - 7)\)  
21. \((x - 5)(x - 7)\)

22. \((3x - 5)(2x + 4)\)  
23. \((-x + 5)(3x + 8)\)

24. \((2x - 5)(2x + 5)\)  
25. \((3x + 7)(3x - 7)\)

26. \((3x^2 - 5)(x - 6)\)  
27. \((8x - 2)(-4x^2 - 7)\)

28. \((2x - 7)(x^2 - 3x + 9)\)  
29. \((5x + 3)(2x^2 - x + 5)\)

30. \((x - 1)(x - 1)\)  
31. \((x + 1)^2\)

32. \((2x - 3)^2\)  
33. \((5x + 4)^2\)

34. \(2x(2x - 1)(3x + 2)\)  
35. \(4x(3x - 2)(2x + 3)\)

36. \((x - 5)(2x - 1)(3x + 2)\)  
37. \((2x + 1)(3x + 1)(x - 2)\)

38. \((x - 1)(x + 1)\)  
39. \((x - 1)(x^2 + x + 1)\)

40. \((x - 1)(x^2 + x^2 + x + 1)\)

41. \((x - 1)(x^4 + x^3 + x^2 + x + 1)\)

42. What do you think the answer is for 
\((x - 1)(x^{50} + x^{49} + \ldots + x^2 + x + 1)\)?
4–5 Special Binomial Operations

A binomial is a two-term polynomial. Usually we use the term for expressions of the form \( ax + b \).

A trinomial is a three term polynomial. Usually we use the term for expressions of the form \( ax^2 + bx + c \).

The product of two binomials is a trinomial.

\((#x + #)(#x + #) = #x^2 + #x + #\)

F: To get the \( x^2 \)-term, multiply the two Front \( x \)-terms of the binomials.

OI: To get the \( x \)-term, multiply the Outer and Inner pairs and combine the results.

L: To get the constant term, multiply the two Last constant terms.

This is called the FOIL method.

The FOIL method speeds up the multiplication of above binomial products and this will come in handy later.

---

**Special Binomial Operations**

Example A. Multiply using FOIL method.

a. \((x + 3)(x - 4) = x^2 - x - 12\)

   The last terms: -12

b. \((3x + 4)(-2x + 5) = -6x^2 + 7x + 20\)

   The last terms: 20

Expanding the negative of the binomial product requires extra care. One way to do this is to insert a set of "[ ]" around the product.

Example B. Expand.

a. \(-[(3x - 4)(x + 5)]\) Insert [ ]

   \(-[3x^2 + 15x - 4x - 20]\) Expand

   \(-[3x^2 + 11x - 20]\) Remove [ ] and change all the signs.

   \(-3x^2 - 11x + 20\)

The key here is that *all three terms change signs.*
Special Binomial Operations

Another way to do this is to distribute the negative sign into the first binomial then FOIL.

Example C. Expand.

a. \(- (3x - 4)(x + 5)\)
   \[= (-3x + 4)(x + 5)\]  \[\text{Distribute the sign.}\]
   \[= - 3x^2 - 15x + 4x + 20\]  \[\text{Expand}\]
   \[= - 3x^2 - 11x + 20\]

Below we present both versions of the algebra for simplifying the differences of two products of binomials.

Example D. Expand and simplify.

a. \((2x - 5)(x + 3) - [(3x - 4)(x + 5)]\)  \[\text{Insert brackets}\]
   \[= 2x^2 + x - 15 - [3x^2 + 11x - 20]\]  \[\text{Expand}\]
   \[= 2x^2 + x - 15 - 3x^2 - 11x + 20\]  \[\text{Remove brackets and combine}\]
   \[= -x^2 - 10x + 5\]

b. Expand and simplify.
   \((2x - 5)(x + 3) - (3x - 4)(x + 5)\)
   \[= (2x - 5)(x + 3) + (-3x + 4)(x + 5)\]  \[\text{Distribute the “-” sign}\]
   \[= 2x^2 + 6x - 5x - 15 - 3x^2 - 15x + 4x + 20\]  \[\text{Expand}\]
   \[= 2x^2 + x - 15 - 3x^2 - 11x + 20\]
   \[= -x^2 - 10x + 5\]
Special Binomial Operations

If the binomials are in x and y, then the products consist of the \(x^2\), xy and \(y^2\) terms. That is,

\[(\#x + \#y)(\#x + \#y) = \#x^2 + \#xy + \#y^2\]

The FOIL method is still applicable in this case.

Example E. Expand.

\[(3x - 4y)(x + 5y)\]

\[= 3x^2 + 15xy - 4yx - 20y^2 = 3x^2 + 11xy - 20y^2\]

\[\uparrow \quad \uparrow \quad \uparrow\]

F OI L

---

Special Binomial Operations

Exercise A. Expand by FOIL method first. Then do them by inspection.

1. \((x + 5)(x + 7)\)
2. \((x - 5)(x + 7)\)
3. \((x + 5)(x - 7)\)
4. \((x - 5)(x - 7)\)
5. \((3x - 5)(2x + 4)\)
6. \((-x + 5)(3x + 8)\)
7. \((2x - 5)(2x + 5)\)
8. \((3x + 7)(3x - 7)\)
9. \((-3x + 7)(4x + 3)\)
10. \((-5x + 3)(3x - 4)\)
11. \((2x - 5)(2x + 5)\)
12. \((3x + 7)(3x - 7)\)
13. \((9x + 4)(5x - 2)\)
14. \((-5x + 3)(-3x + 1)\)
15. \((5x - 1)(4x - 3)\)
16. \((6x - 5)(-2x + 7)\)
17. \((x + 5y)(x - 7y)\)
18. \((x - 5y)(x - 7y)\)
19. \((3x + 7y)(3x - 7y)\)
20. \((-5x + 3y)(-3x + y)\)

B. Expand and simplify.

21. \(-(2x - 5)(x + 3)\)
22. \(-(6x - 1)(3x - 4)\)
23. \(-(8x - 3)(2x + 1)\)
24. \(-(3x - 4)(4x - 3)\)
Special Binomial Operations

C. Expand and simplify.
25. \((3x - 4)(x + 5) + (2x - 5)(x + 3)\)
26. \((4x - 1)(2x - 5) + (x + 5)(x + 3)\)
27. \((5x - 3)(x + 3) + (x + 5)(2x - 5)\)
28. \((3x - 4)(x + 5) - (2x - 5)(x + 3)\)
29. \((4x - 4)(2x - 5) - (x + 5)(x + 3)\)
30. \((5x - 3)(x + 3) - (x + 5)(2x - 5)\)
31. \((2x - 7)(2x - 5) - (3x - 1)(2x + 3)\)
32. \((3x - 1)(x - 7) - (x - 7)(3x + 1)\)
33. \((2x - 3)(4x + 3) - (x + 2)(6x - 5)\)
34. \((2x - 5)^2 - (3x - 1)^2\)
35. \((x - 7)^2 - (2x + 3)^2\)
36. \((4x + 3)^2 - (6x - 5)^2\)
4–6 Multiplication Formulas

The most important product-formulas are:

(The Squares) \[(A + B)(A + B) = (A + B)^2\]
\[(A - B)(A - B) = (A - B)^2\]

(The Conjugates Product) \[(A + B)(A - B)\]

The Conjugates Product and the Difference of Squares
The two binomials \((A + B)\) and \((A - B)\) are said to be the conjugate of each other.
For example, the conjugate of \((3x + 2)\) is \((3x - 2)\),
and the conjugate of \((2ab - c)\) is \((2ab + c)\).

The Conjugate Product:
\[(A + B)(A - B) = A^2 - B^2\]


Note: The conjugate of \((3x + 2)\) or \((3x - 2)\) is different from the opposite of \((3x + 2)\) which is \((-3x - 2)\).

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Multiplication Formulas

Here are some examples of squaring: \((3x)^2 = 9x^2\),
\((2xy)^2 = 4x^2y^2\), and \((5z)^2 = 25z^2\).

Example A. Expand using the formula.

a. \((3x + 2)(3x - 2) = (3x)^2 - (2)^2 = 9x^2 - 4\)
   \[
   (A + B)(A - B) = A^2 - B^2
   \]

b. \((2xy - 5z^2)(2xy + 5z^2)\)
   \[
   = (2xy)^2 - (5z^2)^2
   = 4x^2y^2 - 25z
   \]

II. Square Formulas

\[\begin{align*}
(A + B)^2 &= A^2 + 2AB + B^2 \\
(A - B)^2 &= A^2 - 2AB + B^2 
\end{align*}\]

Check this by multiplying,
\[(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2\]
We say that “\((A + B)^2\) is \(A^2\), \(B^2\), plus twice \(A*B\),”
and “\((A - B)^2\) is \(A^2\), \(B^2\), minus twice \(A*B\).”
Multiplication Formulas

Example B. Expand using the formula.

a. \((3x + 4)^2 = (3x)^2 + 2(3x)(4) + 4^2 = 9x^2 + 24x + 16\)

\((A + B)^2 = A^2 + 2AB + B^2\)

b. \((3a - 5b)^2 = (3a)^2 - 2(3a)(5b) + (5b)^2\)

= \(9a^2 - 30ab + 25b^2\)

III. Some Applications of the Formulas

We can use the above formulas to help us multiply numbers.

Example C. Calculate. Use the conjugate formula.

a. \(51 \times 49 = (50 + 1)(50 - 1) = 50^2 - 1^2 = 2,500 - 1 = 2,499\)

b. \(52 \times 48 = (50 + 2)(50 - 2) = 50^2 - 2^2 = 2,500 - 4 = 2,496\)

c. \(63 \times 57 = (60 + 3)(60 - 3) = 60^2 - 3^2 = 3,600 - 9 = 3,591\)

The conjugate formula

\((A + B)(A - B) = A^2 - B^2\)

may be used to multiply two numbers of the forms \((A + B)\) and \((A - B)\) where \(A^2\) and \(B^2\) can be calculated easily.

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Multiplication Formulas

The Squaring Formulas.

“(A + B)^2 is A^2, B^2, plus twice A*B”,

“(A – B)^2 is A^2, B^2, minus twice A*B”.

Example D. Calculate. Use the squaring formulas.

a. \(51^2 = (50 + 1)^2 = 50^2 + 1^2 + 2(50)(1)\)

= \(2,500 + 1 + 100\)

= \(2,601\)

b. \(49^2 = (50 - 1)^2 = 50^2 + 1^2 - 2(50)(1)\)

= \(2,500 + 1 - 100\)

= \(2,401\)

c. \((50\frac{1}{2})^2 = (50 + \frac{1}{2})^2 = 50^2 + \frac{1}{2}^2 + 2(\frac{1}{2})(50)\)

= \(2,500 + 1/4 + 50\)

= \(2,550\frac{1}{4}\)

We may take advantage of these formulas for lengthier multiplication.
Example E. Expand using the formula.

\[(3x - 2y + 4)(3x - 2y - 4)\]

\[= [(3x - 2y) + 4][(3x - 2y) - 4] \quad \text{group into conjugates}\]
\[= (3x - 2y)^2 - 4^2 \quad \text{difference of squares}\]
\[= (3x)^2 - 2(3x)(2y) + (2y)^2 - 4^2\]
\[= 9x^2 + 12xy + 4y^2 - 16\]
4-7 Division of Polynomials

In arithmetic we +, -, and * whole numbers and obtaining whole number answers.

Likewise in algebra, we +, -, and * polynomials and obtaining polynomial answers.

Similarly there is a long division algorithm for polynomials that corresponds to dividing numbers.

Just as the case for dividing numbers, we will do the case where the division yields no remainder.

We will come back for the case that yields a remainder after we learn how to do fractional algebra so we may check our answers.

We start by reviewing the long division for the whole numbers.

**The Vertical Format**

We demonstrate the vertical long-division of numbers below.

Example A. Divide $78 \div 2$

We enter the quotient on top if possible:

```
   39
  2 78
```

```
2 x 3
```

```
```

```
```

So the remainder $R$ is 0 and we have that $78 \div 2 = 39$ evenly and that $2 \times 39 = 78$.

**Division**

Steps.

i. (Front-in Back-out)

Put the problem in a scaffold format with the front-number, the dividend, inside the scaffold, and the back-number, the divisor, outside the scaffold.

ii. Enter the quotient on top, multiply the quotient back into the scaffold and subtract the results from the dividend, bring down the unused digits, if any. This is the new dividend. Enter a quotient and repeat step ii.

iii. If the new dividend is not enough to be divided by the divisor, STOP! This is the remainder $R$. 
The Long Division

Numerator $\rightarrow N(x)$
Denominator $\rightarrow D(x)$

Set up for the division $\frac{N(x)}{D(x)}$ the same way for dividing numbers.

Example B. Divide $\frac{2x^2 - 3x + 20}{x - 4}$ using long division.

1. Set up the long division, $N(x)$ inside, $D(x)$ outside.
2. Enter on top the quotient of the leading terms.
3. Multiply this quotient to the divisor and subtract the result from the dividend.
4. This difference is the new dividend, repeat steps 2 and 3.
5. Stop when the degree of the new dividend is smaller than the degree of the divisor, i.e. no more quotient is possible.

The remainder is 0, so that $(x - 4)(2x + 5) = 2x^2 - 3x + 20$

The Long Division

(Long Division Theorem)
If $Q(x)$ is the quotient of $\frac{N(x)}{D(x)}$ with the remainder 0, then $Q(x)D(x) = N(x)$.

Example C. Divide using the long division $\frac{x^3 - 2x^2 + x - 2}{x^2 + 1}$

$x - 2$
$x^2 + 1 | x^3 - 2x^2 + x - 2$
$- \quad \quad \quad + x$
$\quad \quad \quad - 2x^2$
$\quad \quad \quad \quad + 2$
$\quad \quad \quad \quad \quad 0$ (the remainder)

Hence $\frac{x^3 - 2x^2 + x - 2}{x^2 + 1} = x - 2$

Check that: $(x^2 + 1)(x - 2) = x^3 - 2x^2 + x - 2$
The Long Division

A variation on this method to embed the subtraction into the process by using the negative the divisor in the set up to avoid mistakes.

To divide \( \frac{x^3 - 2x^2 + x - 2}{x^2 + 1} \) using the long division, set up the division by changing \( x^2 + 1 \) to \( -x^2 - 1 \) to incorporate the subtraction operation.

\[
x^2 + 1 \longmapsto -x^2 - 1 \overbrace{x^3 - 2x^2 + x - 2}^{+)} -x^3 - x
\]
\[
+ \quad 0 - 2x^2 \quad 0 - 2
\]
\[
+ \quad 2x^2 + 2
\]
\[
0
\]

The Long Division

Divide using long division (there is no remainder) and check your answers.

1. \( \frac{x^2 - x - 2}{x + 1} \)  
2. \( \frac{x^2 - x - 2}{x - 2} \)  
3. \( \frac{x^2 + x - 2}{x + 2} \)  
4. \( \frac{x^2 - x - 2}{x - 1} \)  
5. \( \frac{-x^2 + 2x + 3}{x + 1} \)  
6. \( \frac{x^2 + 2x - 3}{1 - x} \)  
7. \( \frac{x^2 - 3x + 2}{2 - x} \)  
8. \( \frac{x^2 - 5x + 6}{x - 6} \)  
9. \( \frac{x^2 + 2x - 8}{x - 2} \)  
10. \( \frac{x^2 - 5x + 6}{x - 2} \)  
11. \( \frac{-x^2 - x + 2}{x - 2} \)  
12. \( \frac{x^2 + 5x - 6}{x - 1} \)  
13. \( \frac{x^2 - 9}{x + 3} \)  
14. \( \frac{1 - x^2}{x - 1} \)  
15. \( \frac{-6x^2 + 7x - 20}{2x + 5} \)  
16. \( \frac{2x^2 - 5x + 2}{2x - 1} \)  
17. \( \frac{2x^2 - 5x - 3}{2x + 1} \)  
18. \( \frac{6x^3 - 5x^2 + x + 12}{x + 1} \)  
19. \( \frac{3x^3 - 5x^2 + 2}{x - 1} \)  
20. \( \frac{3x^3 + 2x^2 + 3x + 2}{x^2 + 1} \)  
21. \( \frac{ax^3 + bx^2 + ax + b}{x^2 + 1} \)
5-1 Factoring Out GCF
There are three boxes A, B, and C as shown here.

We are to take items from all three boxes and the items taken from the boxes must be the same. For example, we may take two apples from each box, or three bananas from each box, or two bananas and two carrot.

A group of items which may be taken from each of the three boxes is a group of common items.

In this case the largest group of items which may be taken from each of the three boxes consists of.

We define the “greatest common factor” in a similar way.

Factoring Out GCF

To factor means to rewrite a quantity as a product (without using 1). A quantity that can’t be written as a product besides as \( 1 \times x \) is said to be prime. To factor completely means each factor in the product is prime.

Example A. Factor 12 completely.

\[
12 = 3 \times 4 = 3 \times 2 \times 2
\]

not prime factored completely

A common factor of two or more quantities is a factor that belongs to all the quantities.

Example B.

a. Since \( 6 = 2 \times 3, \ 15 = 3 \times 5, \ 3 \) is a common factor.
b. The common factors of \( 4ab, \ 6a \) are \( 2, \ a, \ 2a \).
c. The common factors of \( 6xy^2, \ 15x^2y \) are \( x, \ y^2, \ xy^2 \). ..

The common factor may be a formula in in parenthesis:
d. The common factor of \( a(x+y), \ b(x+y) \) is \( (x+y) \).
Factoring Out GCF

The greatest common factor (GCF) is the common factor that has the largest coefficient and highest degree of each factor among all common factors.

Example C. Find the GCF of the given quantities.

a. GCF{24, 36} = 12.

b. GCF{4ab, 6a} = 2a.

c. GCF {6xy^2, 15 x^2y^2} = 3xy^2.

d. GCF{x^3y^5, x^4y^4} = x^3y^4.

The Extraction Law

Distributive law interpreted backward gives the Extraction Law, that is, common factors may be extracted from sums or differences.

\[ AB ± AC \rightarrow A(B±C) \]

This procedure is also called “factoring out a common factor”. To factor, the first step always is to factor out the GCF, then factor the “left over” if it’s needed.

Factoring Out GCF

Example D. Factor out the GCF.

a. \( xy - 4y = y(x - 4) \) or \( (x - 4)y \)
   (the GCF is \( y \))

b. \( 4ab + 6a = 2a(2b) + 2a(3) = 2a(2b + 3) \)
   (the GCF is \( 2a \))

c. \( 12x^2y^3 + 6x^2y^2 = 6x^2y^2(2y) + 6x^2y^2(1) = 6x^2y^2(2y + 1) \)
   (the GCF is \( 6x^2y^2 \))

We may pull out common factors that are (\( a \))’s.

Example E. Factor

a. \( a(x + y) - 4(x + y) \)
   Pull out the common factor \( (x + y) \)
   \[ a(x + y) - 4(x + y) = (a - 4)(x + y) \]

b. \( (2x - 3)3x - 2(2x - 3) \)
   Pull out the common factor \( (2x - 3) \),
   \[ (2x - 3)3x - 2(2x - 3) = (2x - 3)(3x - 2) \]
Factor by Grouping

There are special four-term formulas where we have to separate the terms into two pairs, factor out each pair’s GCF to reveal a common parenthesis—factor, then we pull out this common parenthesis.

Example F. Factor by pulling out twice.

a. \(3x - 3y + ax - ay\)  
   Group them into two groups.
   \[= (3x - 3y) + (ax - ay)\]  
   Factor out the GCF of each group.
   \[= 3(x - y) + a(x - y)\]  
   Pull the factor \((x - y)\) again.
   \[= (3 + a)(x - y)\]

We may need to pull out the negative sign  
e.g. writing \(-4x + 10\) as \(-(2x - 5)\),  
in the expression to reveal the common factor.

b. \(y(2x - 5) - 4x + 10\)  
   \[= y(2x - 5) - 2(2x - 5)\]  
   \[= (y - 2)(2x - 5)\]

Factor by Grouping

**Trinomials (three-term)** are polynomials of the form \(ax^2 + bx + c\) where \(a\), \(b\), and \(c\) are numbers.

The product of two binomials is a trinomials:

\[(\#x + \#)(\#x + \#) \rightarrow ax^2 + bx + c\]

For example \((x + 2)(2x + 3) = 2x^2 + 7x + 6\).

To factor a trinomial means to undo the multiplication and write the trinomial as a product of two binomials, if possible. 
\[ax^2 + bx + c \rightarrow (\#x + \#)(\#x + \#)\]

Example G.

Factor the the trinomial \(x^2 - 3x + 2\) by grouping given that \(x^2 - 3x + 2 = x^2 - 2x - x + 2\)

Group \(x^2 - 2x - x + 2\) into two groups.
\[x^2 - 2x - x + 2\]  
\[= (x^2 - 2x) + (-x + 2)\]  
Factor out the GCF of each group.
\[= x(x - 2) - 1(x - 2)\]  
Pull the factor \((x - 2)\) again.
\[= (x - 2)(x - 1)\]  
We will use grouping as the default method for factoring trinomials.
Factoring Out GCF

Exercise A. Find the GCF of the listed quantities.
1. \{4, 6\}  2. \{12, 18\}  3. \{32, 20, 12\}  4. \{25, 20, 30\}
5. \{4x, 6x^2\}  6. \{12x^2y, 18xy^2\}
7. \{32A^2B^3, 20A^3B^3, 12A^3B^3\}
8. \{25x^7y^6z^6, 20y^7z^5x^6, 30z^8x^5y^6\}

B. Factor out the GCF.
9. \(4 - 6y\)  10. \(12x + 16y\)  11. \(32A + 20B - 12C\)
12. \(25x + 20y - 30\)  13. \(-4x + 6x^2\)
14. \(-12x^2y - 18xy^2\)  15. \(32A^2B^3 - 20A^3B^3 - 12A^2B^3\)
16. \(25x^7y^6z^6 - 20y^7z^5x^6 + 30z^8x^5y^6\)
17. \(4x^4 - 8x^3 + 2x^2\)  18. \(20x^4 - 5x^2\)
19. \(x(x - 2) + 3(x - 2)\)  20. \(4x(2x - 3) - 5(2x - 3)\)

C. Factor out the “-”.
21. \(-2y + 4\)  22. \(-3x + 18\)  23. \(-5x + 15\)  24. \(-8x + 16\)

Factoring Out GCF

D. Factor, use grouping if it’s necessary.
25. \(y^2 - 2y + 3y - 6\)  26. \(x^2 + 3x + 6x + 18\)
27. \(y^2 - 2y - 3y + 6\)  28. \(x^2 + 3x - 6x - 18\)
29. \(y^2 - y + 4y - 4\)  30. \(x^2 - 5x - 2x + 10\)
31. \(2y^2 - y - 6y + 3\)  32. \(3x^2 + 2x - 6x - 4\)
33. \(4x^2 + 6x - 5x - 9\)  34. \(-3x^2 + 4x - 6x + 8\)
35. \(-5y^2 + 10y - 3y + 6\)  36. \(-x^2 + 3x - 7x + 21\)
37. \(2y^2 - xy - 6xy + 3x^2\)  38. \(3x^2 + 2xy - 6xy - 4y^2\)
39. \(-5x^2 + 2xy - 20xy + 8y^2\)  40. \(-14x^2 + 21xy - 8xy + 12y^2\)
5–2 Factoring Trinomials and Making Lists

For our discussions, trinomials (three-term) in $x$ are polynomials of the form $ax^2 + bx + c$ where $a\neq 0$, $b$, and $c$ are numbers.

In general, we have $(\#x + \#)(\#x + \#) \rightarrow ax^2 + bx + c$.

For example,

$(x + 2)(x + 1) \rightarrow x^2 + 3x + 2$ with $a = 1$, $b = 3$, and $c = 2$.

Hence, "to factor a trinomial" means to write the trinomial as a product of two binomials, that is, to convert

$ax^2 + bx + c \rightarrow (\#x + \#)(\#x + \#)$

**The Basic Fact About Factoring Trinomials:**

There are two types of trinomials,

I. the ones that are *factorable* such as

$x^2 + 3x + 2 \rightarrow (x + 2)(x + 1)$

II. the ones that are *prime* or no factorable, such as

$x^2 + 2x + 3 \rightarrow (\#x + \#)(\#x + \#)$ (Not possible!)

Our jobs are to determine which trinomials:

1. are factorable and factor them,
2. are prime so we won’t waste time on trying to factor them.

---

**Factoring Trinomials and Making Lists**

**The ac-Method**

A table like the ones above can be made from a given trinomial and the ac–method uses the table to check if the given trinomial is factorable or prime.

I. If we find the $u$ and $v$ that fit the table then it is factorable, and we may use the grouping method, with the found $u$ and $v$, to factor the trinomial.

II. If the table is impossible to do, then the trinomial is prime.

Here is an example of factoring a trinomial by grouping.

**Example B.** Factor $x^2 – x – 6$ by grouping.

\[x^2 - x - 6\]

write \(-x\) as \(-3x + 2x\)

\[= x^2 - 3x + 2x - 6\]

put the four terms into two pairs

\[= (x^2 - 3x) + (2x - 6)\]

take out the GCF of each pair

\[= x(x - 3) + 2(x - 3)\]

take out the common \((x - 3)\)

\[= (x - 3)(x + 2)\]

Let’s see how the X–table is made from a trinomial.
Factoring Trinomials and Making Lists

ac-Method: Given the trinomial \( ax^2 + bx + c \)
with no common factor, its ac-table is:
i.e. \( ac \) at the top, and \( b \) at the bottom,
and we are to find \( u \) and \( v \) such that
\[
\begin{align*}
uv &= ac \\
u + v &= b
\end{align*}
\]

1. If \( u \) and \( v \) are found (so \( u + v = b \)),
write \( ax^2 + bx + c \) as \( ax^2 + ux + vx + c \).
then factor \((ax^2 + ux) + (vx + c)\) by the grouping method.

   In example B, the ac-table for \( 1x^2 - x - 6 \) is:
   We found \(-3, 2\) fit the table, so we write
   \( x^2 - x - 6 \) as \( x^2 - 3x + 2x - 6 \) and by grouping
   \[
   (x^2 - 3x) + (2x - 6)
   = x(x - 3) + 2(x - 3)
   = (x - 3)(x + 2)
   \]

Factoring Trinomials and Making Lists

Example C. Factor \( 3x^2 - 4x - 20 \) using the \( ac \)-method.

We have that \( a = 3, c = -20 \) so \( ac = 3(-20) = -60 \).
\( b = -4 \) and the ac-table is:

\[
\begin{array}{cc}
-60 & 6 \\
-10 & 6
\end{array}
\]

We need two numbers \( u \) and \( v \) such that
\( uv = -60 \) and \( u + v = -4 \).

By trial and error we see that \(-6 \) and \(-10 \) is the
solution so we may factor the trinomial by grouping.

Using \(-6 \) and \(-10 \), write \( 3x^2 - 4x - 20 \) as \( 3x^2 + 6x - 10x - 20 \)
\[
= (3x^2 + 6x) + (-10x - 20) \quad \text{put in two groups}
= 3(x + 2) - 10(x + 2) \quad \text{pull out common factor}
= (3x - 10)(x + 2) \quad \text{pull out common factor}
\]
Hence \( 3x^2 - 4x - 20 = (3x - 10)(x + 2) \)

If the trinomial is prime then we have to justify it's prime by
showing that no such \( u \) and \( v \) exist by listing all the possible
\( u \)'s and \( v \)'s such that \( uv = ac \) in the table to demonstrate that
none of them fits the condition \( u + v = b \).
Factoring Trinomials and Making Lists

Example D. Factor $3x^2 - 6x - 20$ if possible.

If it's prime, justify that.

$a = 3, c = -20$, hence $ac = 3(-20) = -60$,
with $b = -6$, we have the ac-table.

We want two numbers $u$ and $v$ such that $uv = -60$ and $u + v = -6$.

After failing to guess two such numbers,
we check to see if it's prime by listing in order all positive $u$'s and $v$'s where $uv = 60$ as shown.

By the table, we see that there are no $u$ and $v$ such that $(\pm) u$ and $v$ combine to be $-6$.
Hence $3x^2 - 6x - 20$ is prime.

Factoring By Trial and Error

Finally for some trinomials, such as when $a = 1$ or $x^2 + bx + c$,
it's easier to guess directly because it must factor into the form $(x \pm u)(x \pm v)$ if it's factorable.

Example E.

a. Factor $x^2 + 5x + 6$

We want $(x + u)(x + v) = x^2 + 5x + 6$, so we need $u$ and $v$ where $uv = 6$ and $u + v = 5$.

Since $6 = (1)(6) = (2)(3) = (-1)(-6) = (-2)(-3)$ and $2x + 3x = 5x$,
so $x^2 + 5x + 6 = (x + 2)(x + 3)$

b. Factor $x^2 - 5x + 6$

We want $(x + u)(x + v) = x^2 - 5x + 6$, so we need $u$ and $v$ where $uv = 6$ and $u + v = -5$.

Since $6 = (1)(6) = (2)(3) = (-1)(-6) = (-2)(-3)$ and $-2 - 3 = -5$,
so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

c. Factor $x^2 + 5x - 6$

We want $(x + u)(x + v) = x^2 + 5x - 6$, so we need $uv = -6$ and $u + v = 5$.

Since $-6 = (-1)(6) = (1)(-6) = (-2)(3) = (2)(-3)$ and $-1 + 6 = 5$,
so $x^2 + 5x - 6 = (x - 1)(x + 6)$. 
By Trial and Error
This reversed-FOIL method (by trial and error) is useful when
the numbers involved can only be factored in few options.

Example F. Factor \(3x^2 + 5x + 2\).
The only way to get \(3x^2\) is \((3x \pm \#)(1x \pm \#)\).
The \#’s must be 1 and 2 to get the constant term +2.
We need to place 1 and 2 as the \#’s so the product will
yield the correct middle term +5x.
That is, \((3x \pm \#)(1x \pm \#)\) must yields +5x, or that
\[3(\pm \#) + 1(\pm \#) = 5\] where the \#’s are 1 and 2.
Since \(3(1) + 1(2) = 5\), we see that
\(3x^2 + 5x + 2 = (3x + 2)(1x + 1)\).

Factoring Trinomials and Making Lists
Besides the ac–method, here is another method that’s based
on a calculating a number to check if a trinomial is factorable.
Theorem: The trinomial \(ax^2 + bx + c\) is factorable
if \(b^2 - 4ac = 0, 1, 4, 9, 16, 25\) ...i.e. it’s a squared number.
If \(b^2 - 4ac = \) not a squared number, then it’s not factorable.

Example G. Use \(b^2 - 4ac\) to check if the trinomial is factorable.
a. \(3x^2 - 7x + 2\)
\[b^2 - 4ac\] \[a = 3, b = (-7)\] and \(c = 2\)
\[= (-7)^2 - 4(3)(2)\]
\[= 49 - 24\]
\[= 25\] which is a squared number, hence it is factorable.
b. \(3x^2 - 7x - 2\)
\[b^2 - 4ac\] \[a = 3, b = (-7)\] and \(c = (-2)\)
\[= (-7)^2 - 4(3)(-2)\]
\[= 49 + 24\]
\[= 73\] is not a square, hence it is prime.
Factoring Trinomials and Making Lists

Observations About Signs

Given that \( x^2 + bx + c = (x + u)(x + v) \) so that \( uv = c \), we observe the following.

1. If \( c \) is positive, then \( u \) and \( v \) have same sign.
   In particular,
   \[
   \begin{cases} 
   \text{if } b \text{ is also positive, then both are positive.} \\
   \text{if } b \text{ is negative, then both are negative.}
   \end{cases}
   \]
   From the examples above
   \[
   \begin{align*}
   x^2 + 5x + 6 &= (x + 2)(x + 3) \\
   x^2 - 5x + 6 &= (x - 2)(x - 3)
   \end{align*}
   \]

2. If \( c \) is negative, then \( u \) and \( v \) have opposite signs. The one with larger absolute value has the same sign as \( b \).
   From the example above
   \[
   x^2 - 5x - 6 = (x - 6)(x + 1)
   \]

Factoring Trinomials and Making Lists

Exercise A. Use the ac–method, factor the trinomial or demonstrate that it’s not factorable.

1. \( 3x^2 - x - 2 \)  
2. \( 3x^2 + x - 2 \)  
3. \( 3x^2 - 2x - 1 \)  
4. \( 3x^2 + 2x - 1 \)  
5. \( 2x^2 - 3x + 1 \)  
6. \( 2x^2 + 3x - 1 \)  
7. \( 2x^2 + 3x - 2 \)  
8. \( 2x^2 - 3x - 2 \)  
9. \( 5x^2 - 3x - 2 \)  
10. \( 5x^2 + 9x - 2 \)  
11. \( 3x^2 + 5x + 2 \)  
12. \( 3x^2 - 5x + 2 \)  
13. \( 3x^2 - 5x + 2 \)  
14. \( 6x^2 - 5x - 6 \)  
15. \( 6x^2 + 5x - 6 \)  
16. \( 6x^2 - x - 2 \)  
17. \( 6x^2 - 13x + 2 \)  
18. \( 6x^2 - 13x + 2 \)  
19. \( 6x^2 + 7x + 2 \)  
20. \( 6x^2 - 7x + 2 \)  
21. \( 6x^2 - 13x + 6 \)  
22. \( 6x^2 + 13x + 6 \)  
23. \( 6x^2 - 5x - 4 \)  
24. \( 6x^2 - 13x + 8 \)  
25. \( 6x^2 - 13x - 8 \)  
26. \( 4x^2 - 9 \)  
27. \( 25x^2 - 4 \)  
28. \( 4x^2 + 9 \)  
29. \( 25x^2 + 9 \)

Exercise B. Factor. Factor out the GCF, the “\( - \)”, and arrange the terms in order first.

30. \( -6x^2 - 5xy + 6y^2 \)  
31. \( -3x^2 + 2x - x \)  
32. \( -6x^3 - x^2 + 2x \)  
33. \( -15x^3 - 25x^2 - 10x \)  
34. \( 12x^2y^2 - 14x^2y^2 + 4xy^2 \)
Factoring Trinomials and Making Lists

C. Factor. Factor out the GCF, the “-”, and arrange the terms in order first.
34. \(-xy^2 + 4yx + 5y\) 35. \(-3x^3 - 30x^2 - 48x\)
36. \(-2x^3 + 20x^2 - 24x\) 37. \(-x^2 + 11xy + 24y^2\)
38. \(x^3 - 6x^2 + 36x^2\) 39. \(-x^2 + 9xy + 36y^2\)
40. \(4x^2 - 44xy + 96y^2\)

D. Factor. If not possible, state so.
41. \(x^2 + 1\) 42. \(x^2 + 4\) 43. \(x^2 + 9\) 44. \(4x^2 + 25\)
44. What can you conclude from 41–43?
5–3 Factoring Formulas and Substitution

We evaluated expressions with numerical inputs.
Evaluating \( A^2 - B^2 \) with \( A = 2, B = 1 \) as inputs, the output is 3.
We extend this procedure to using expressions as inputs and say that we "substitute the variables with the expressions...".

Example A. Substitute \( A^2 - B^2 \) with \( A = x + 2 \) and \( B = 2x \).
Expand and simplify the result.
Replace \( A \) by \((x + 2)\), and \( B \) by \((2x)\) we have
\[
(x + 2)^2 - (2x)^2
= x^2 + 4x + 4 - 4x^2
= -3x^2 + 4x + 4
\]
We also say that we "plug in" \((x + 2)\) for \( A \)

We make sure to encase the inputs in the \((\ )\)’s

Reverse the procedure. Sometime the inputs can be easily identified when the formula and the outcome are given.

Example B. Evaluating \( A^2 - B^2 \) and we obtain
a. \( 4x^2 - 9y^2 \), what are the \( A \) and \( B \)?
Matching the \( A^2 = 4x^2 \), we’re asking \(( ? )^2 = 4x^2 \), so \( A = (2x) \).
Similarly we conclude that \( B = (3y) \).

Factoring Formulas and Substitution

b. \( 25x^2y^4 - 1 \), what are the \( A \) and \( B \)?
Matching the \( A^2 = 25x^2y^4 \) and \( B^2 = 1 \), we have that
\( A = 5xy^2 \) and \( B = 1 \).
c. \( x^2 + 2xy + y^2 - 9 \), what are the \( A \) and \( B \)?
Factor \( x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2 \),
so \( x^2 + 2xy + y^2 - 9 = (x + y)^2 - 3^2 \).
Hence \( A = (x + y) \) and \( B = 3 \).

Example C. Evaluating \( A^3 - B^3 \) with inputs \( A \) and \( B \)
a. we obtain \( x^3 - 125 \), what are the \( A \) and \( B \)?
Matching \( A^3 = x^3 \), we have \( A = x \).
Similarly we’ve \( B^3 = 125 \) so \( B = 5 \).
b. we obtain \( 8y^3 - (2 - y)^3 \), what are the \( A \) and \( B \)?
Matching the \( A^3 = 8y^3 \), we have \( A = 2y \).
By inspection, we have \( B = (2 - y) \).
Recall the conjugate product formula
\[
(A - B)(A + B) = A^2 - B^2
\]
Factoring Formulas and Substitution

The reverse of the conjugate product formula is the factoring formula for difference of squares:

\[ A^2 - B^2 = (A - B)(A + B) \]

As suggested by the name, we use this formula when the expression is the subtraction of two perfect-square terms.

Example D.

a. \( x^2 - 25 \)
   \[ = (x)^2 - (5)^2 \]
   \[ = (x + 5)(x - 5) \]

b. \( 4x^2 - 9y^2 \)
   \[ = (2x)^2 - (3y)^2 \]
   \[ = (2x - 3y)(2x + 3y) \]

c. \( 25A^2B^4 - 1 \)
   \[ = (5AB)^2 - (1)^2 \]
   \[ = (5AB - 1)(5AB + 1) \]

d. \( (A + 2B)^2 - 16 \)
   \[ = (A + 2B)^2 - 4^2 \]
   \[ = ((A + 2B) - 4)((A + 2B) + 4) \]
   \[ = (A + 2B - 4)(A + 2B + 4) \]

**Caution:** The expressions of two squares of the same sign: \( A^2 + B^2 \) or \( -A^2 - B^2 \) are prime. For example, \( -9 - 4x^2 \) is not factorable.

Sum and Difference of Cubes

Both the difference and the sum of cubes may be factored. If we try to factor \( A^3 - B^3 \), we might guess that

\[ A^3 - B^3 = (A - B)(A^2 + 2) = A^3 - BA^2 + AB^2 - B^3 \]

But this does not work because we have the inner and outer products \( -BA^2 \) and \( AB^2 \) remaining in the expansion as shown here.
Factoring Formulas and Substitution

But if we put in the adjustment $+AB$, then all the cross-products with $A$ and $B$ are eliminated. That is

$$(A - B)(A^2 + AB + B^2)$$

$$= A^3 + A^2B + AB^2 - BA^2 - B^2A - B^3$$

$$= A^3 - B^3$$

Hence $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

Through a similar argument we have that

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

We write these two formulas together as

$$A^3 \pm B^3 = (A \pm B)(A^2 \mp AB + B^2)$$

Note that the signs are reversed for the adjustments in these formulas.

Factoring Formulas and Substitution

Example E.

a. $8x^3 - 27$ (Note that both 8 and 27 are cubes)

$$= (2x)^3 - (3)^3$$

$$= (2x - 3)((2x)^2 + (2x)(3) + (3)^2)$$

$$= (2x - 3)(4x^2 + 6x + 9)$$

b. $64A^3 + 125$ (both 64 and 125 are cubes)

$$= (4A)^3 + (5)^3$$

$$= (4A + 5)((4A)^2 - (4A)(5) + (5)^2)$$

$$= (4A + 5)(16A^2 - 20A + 25)$$

Summary on Factoring

I. Always factor out the GCF first.

II. Make sure the leading term is positive and all terms are arranged in order.

III. If left with a trinomial, use reverse-FOIL or ac-method.

IV. If it is a two-term expression, try factoring by formula.

V. If it is a four term expression, try the grouping-method.
Example F. Factor completely.

A. $20A^3 - 45A$
   Take out the common factor
   $= 5A(4A^2 - 9)$
   Difference of square
   $= 5A(2A - 3)(2A + 3)$

B. $-20A^3 + 45A^2 - 10A$
   Take out the common factor and -1
   $= -5A(4A^2 - 9A + 2)$
   $= -5A(4A - 1)(A - 2)$

---

**Summary on Factoring**

Exercise A. Simplify with the given substitution. Simplify your answers when possible.

For 1-8, Simplify $A^2 - B^2$ with the given substitution.

1. $A = x$, $B = 1$
2. $A = 3$, $B = 2y$
3. $A = 2xy$, $B = 1$
4. $A = x + y$, $B = 2$
5. $A = (x - 3)$, $B = x$
6. $A = 2x$, $B = (x - 1)$
7. $A = (x - 3)$, $B = (x - 1)$
8. $A = (2x + 1)$, $B = (2x - 1)$

For 9-14, simplify $A^3 - B^3$ with the given substitution.

9. $A = x$, $B = 2$
10. $A = 3x$, $B = 2$
11. $A = 2x$, $B = 3y$
12. $A = 1$, $B = 2xy$
13. $A = 4x$, $B = 5$
14. $A = 6x$, $B = 5y$

15. Simplify $x^2 - 2x + 10$ with the substitution $x = (2 + h)$

16. Simplify $x^2 + 3x - 4$ with the substitution $x = (-3 + h)$
Summary on Factoring

B. Identify the substitution for A and B then factor the expression using the pattern $A^2 - B^2 = (A - B)(A + B)$.

17. $x^2 - 1$
18. $9 - 4y^2$
19. $x^2 - 9$
20. $25 - y^2$
21. $25x^2 - 64$
22. $9 - 49y^2$
23. $4x^2y^2 - 1$
24. $96x^2 - 8$
25. $4 - 81x^2y^2$
26. $x^2 + 2xy + y^2 - 4$
27. $50x^2 - 18y^2$
28. $36x^2 - 25x$
29. $8x^2z - 96z$
30. $x^2 - y^2 - 2y - 1$

C. Identify the substitution for A and B then factor the expression using the pattern $A^3 \pm B^3 = (A \pm B)(A^2 \mp AB + B^2)$.

31. $x^3 - 1$
32. $x^3 - 8$
33. $27 - y^3$
34. $x^3 + 8$
35. $64 + y^3$
36. $27 - 8y^3$
37. $x^3 - 125$
38. $27x^3 - 64$
39. $x^3 - 1000$
5–4 Applications of Factoring

The main purposes of factoring an expression \( E \) into a product \( E = AB \) is to utilize the two special properties of multiplication.

i. If the product \( AB = 0 \), then \( A \) or \( B \) must be 0.

ii. The sign of the product can be determined by the signs of \( A \) and \( B \).

Based on these, we may extract a lot more information about a formula from its factored form than its expanded form.

**Solving Equations**

The most important application for factoring is to solve polynomial equations. These are equations of the form

\[
polynomial = polynomial
\]

To solve these equations, we use the following obvious fact.

**The Zero-Product Rule:**

If \( AB = 0 \), then either \( A = 0 \) or \( B = 0 \)

For example, if \( 3x = 0 \), then \( x \) must 0 (because 3 is not 0).

---

**Applications of Factoring**

**Example A.**

a. If \( 3(x - 2) = 0 \), then \( (x - 2) = 0 \), so \( x \) must be 2.

b. If \( (x + 1)(x - 2) = 0 \),

   then either \( (x + 1) = 0 \) or \( (x - 2) = 0 \),
   \[
   x = -1 \quad \text{or} \quad x = 2
   \]

To solve polynomial equation,

1. set one side of the equation to be 0, move all the terms to the other side.
2. factor the polynomial,
3. get the answers.

**Example B. Solve for** \( x \)

a. \( x^2 - 2x - 3 = 0 \)

   Factor

   \[
   (x - 3)(x + 1) = 0
   \]

   Hence \( x - 3 = 0 \) or \( x + 1 = 0 \)

   \[
   x = 3 \quad \text{or} \quad x = -1
   \]

   There are two linear \( x \)-factors. We may extract one answer from each.
Applications of Factoring

b. \[2x(x + 1) = 4x + 3(1 - x)\] Expand
   \[2x^2 + 2x = 4x + 3 - 3x\]
   \[2x^2 + 2x = x + 3\] Set one side 0
   \[2x^2 + 2x - x - 3 = 0\]
   \[2x^3 + x - 3 = 0\] Factor
   \[(2x + 3)(x - 1) = 0\] Get the answers
   \[2x + 3 = 0\] or \[x - 1 = 0\]
   \[2x = -3\] \[x = 1\]
   \[x = -3/2\]

\[x = -3/2\]

c. \[8x(x^2 - 1) = 10x\] Expand
   \[8x^3 - 8x = 10x\]
   \[8x^3 - 8x - 10x = 0\]
   \[8x^3 - 18x = 0\] Factor
   \[2x(2x + 3)(2x - 3) = 0\]
   \[x = 0\] or \[2x + 3 = 0\] or \[2x - 3 = 0\]
   \[2x = -3\] \[2x = 3\]
   \[x = -3/2\] \[x = 3/2\]

Applications of Factoring

Following are two other important applications of the factored forms of polynomials:

- to evaluate polynomials
- determine the sign of the output of a given input

Evaluating Polynomials

Often it is easier to evaluate polynomials in the factored form.

Example C. Evaluate \[x^2 - 2x - 3\] if \[x = 7\]

a. without factoring.
   We get
   \[x^2 - 2x - 3 = (x - 3)(x+1)\]
   \[7^2 - 2(7) - 3 = \]
   \[= 49 - 14 - 3 = 32\]

b. by factoring it first.
   We get
   \[7^2 - 2(7) - 3 = (7 - 3)(7 + 1)\]
   \[= 4(8) = 32\]
Applications of Factoring

Example C. Evaluate \(2x^3 - 5x^2 + 2x\)
for \(x = -2, -1, 3\) by factoring it first.
\[
2x^3 - 5x^2 + 2x = x(2x^2 - 5x + 2)
= x(2x - 1)(x - 2)
\]
For \(x = -2:\)
\[
\]
For \(x = -1:\)
\[
(-1)[2(-1) - 1] [(-1) - 2] = -1 [-3] [-3] = -9
\]
For \(x = 3:\)
\[
\]
Your turn: Double check these answers via the expanded form.

Determine the signs of the outputs

Often we only want to know the sign of the output, i.e.
whether the output is positive or negative. It is easy to do this
using the factored form.

Applications of Factoring

Example D. Determine the outcome is + or − for \(x^2 - 2x - 3\)
if \(x = -3/2, -1/2.\)

Factor \(x^2 - 2x - 3 = (x - 3)(x + 1)\)
Hence for \(x = -3/2:\)
\[
(-3/2 - 3)(-3/2 + 1) \text{ is } (-)(-) = +. \text{ So the outcome is positive.}
\]
And for \(x = -1/2:\)
\[
(-1/2 - 3)(-1/2 + 1) \text{ is } (-)(+) = -.\]
Applications of Factoring

Exercise A. Use the factored form to evaluate the following expressions with the given input values.
1. $x^2 - 3x - 4$, $x = -2, 3, 5$
2. $x^2 - 2x - 15$, $x = -1, 4, 7$
3. $x^2 - 2x - 1$, $x = \frac{1}{2}, -2, -\frac{1}{2}$
4. $x^3 - 2x^2$, $x = -2, 2, 4$
5. $x^3 - 4x^2 - 5x$, $x = -4, 2, 6$
6. $2x^3 - 3x^2 + x$, $x = -3, 3, 5$

B. Determine if the output is positive or negative using the factored form.
7. $x^2 - 3x - 4$, $x = -2\frac{1}{2}, -2/3, 2\frac{1}{2}, 5\frac{1}{4}$
8. $-x^2 + 2x + 8$, $x = -2\frac{1}{2}, -2/3, 2\frac{1}{2}, 5\frac{1}{4}$
9. $x^3 - 2x^2 - 8x$, $x = -4\frac{1}{2}, -3/4, 2\frac{1}{4}, 6\frac{1}{4}$
10. $2x^3 - 3x^2 - 2x$, $x = -2\frac{1}{2}, -3/4, 2\frac{1}{4}, 3\frac{1}{4}$
11. $4x^2 - x^3$, $x = -1.22, 0.87, 3.22, 4.01$
12. $18x - 2x^3$, $x = -4.90, -2.19, 1.53, 3.01$

Applications of Factoring

C. Solve the following equations. Check the answers.
13. $x^2 - 3x - 4 = 0$
14. $x^2 - 2x - 15 = 0$
15. $x^2 + 7x + 12 = 0$
16. $-x^2 - 2x + 8 = 0$
17. $9 - x^2 = 0$
18. $2x^2 - x - 1 = 0$
19. $x^2 = 10$
20. $x(x - 2) = 24$
21. $2x^2 = 3(x + 1) - 1$
22. $x^2 = 4$
23. $8x^2 = 2$
24. $27x^2 - 12 = 0$
25. $2(x - 3) + 4 = 2x - 4$
26. $x(x - 3) + x + 6 = 2x^2 + 3x$
27. $x(x + 4) + 9 = 2(2 - x)$
28. $x^3 - 2x^2 = 0$
29. $x^3 - 2x^2 - 8x = 0$
30. $2x^2(x - 3) = -4x$
31. $4x^2 = x^3$
32. $4x = x^3$
33. $4x^2 = x^4$
34. $7x^3 = -4x^3 - 3x$
35. $5 = (x + 2)(2x + 1)$
36. $(x - 1)^2 = (x + 1)^2 - 4$
37. $(x + 1)^2 = x^2 + (x - 1)^2$
38. $(x + 2)^2 - (x + 1)^2 = x^2$
39. $(x + 3)^2 - (x + 2)^2 = (x + 1)^2$